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Exploring Okun's law asymmetry:  
an endogenous threshold  
LSTR approach

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## Abstract

Using a novel methodology, we offer new evidence that a threshold relationship exists for Okun's law. We use a logistic smoothed transition regression (LSTR) model where possible threshold endogeneity is addressed based on copula theory. We also suggest a new test of the linearity hypothesis against the LSTR model. A combination of structural and policy-related variables accounts for changes (rises) in the Okun's parameter in the US in recent decades. Accordingly, the unemployment gap is increasingly associated with a smaller output gap. Whilst the Great Recession accelerated that rise, the bulk of the change occurred beforehand.

*Keywords:* unemployment, output, asymmetries, logistic transition, endogeneity, copula, Monte Carlo, test for linearity.

*JEL:* C24, C46, E23, E24.

## Non-Technical Summary

Okun's law (OL) refers to a reduced-form relationship between cyclical unemployment and output. It addresses the issue of how much a country's output is "lost" when unemployment exceeds its natural or trend rate. The relationship provides a link between the labor and goods market over the business cycle, and is often considered a key empirical regularity. It is moreover a core part of many macroeconomic models, where the aggregate supply function is derived from combining OL with the Phillips curve which further links to policy trade offs. The relationship also bears implications for macroeconomic policy: it records what rate of growth leads to a reduction in unemployment; plus, it demonstrates that the effectiveness of disinflation policy depends on the responsiveness of unemployment on the rate of output growth.

One common and compelling criticism of the Okun's relationship in the literature, though, is the assumption of linearity. Many studies instead suggest that the relationship is characterized by nonlinearities and asymmetries. A nonlinear asymmetric OL would be an important finding. For instance, it would suggest that the effectiveness (and required 'size') of stabilization policy on the real economy would depend in which 'regime' Okun's relationship then lies. Any non linearity in the relationship would also have implications for macroeconomic projections. Moreover, it may affect other recognised economic relationships such as the price and wage Philips curves.

We use a novel methodology to specifically assess the case for asymmetry. Our econometric framework is an augmented version of the logistic smooth transition regression (LSTR) model. The LSTR model nests the linear and standard threshold specification for "low" and "high" values of its identified regimes. It captures smooth transition across regimes, which may be more reasonable in macroeconomics (compared to abrupt transitions), due to various adjustment mechanisms and frictions.

Notably, though, we depart from the bulk of the literature in that we do not consider the threshold variable to be exogenous. Indeed, in as fundamental and deep-seated a relationship as OL, this is intuitive; if there are asymmetries in OL, then

it is likely that they should arise from the workings of the economy itself – either from the operation of booms and busts, or from policy shifts or more persistent structural changes. In so far as the threshold is endogenous (i.e., contemporaneously correlated with the disturbance term of the regression), our empirical methodology explicitly seeks to capture that endogeneity. Failure to do so leads to biased and inconsistent estimates. To deal with threshold endogeneity, we adjust the mean of the model conditional on the regimes identified by the threshold variable for the endogenous effects based on copula approach.

A combination of structural and policy-related variables accounts for changes (rises) in the Okun's parameter in the US in recent decades. Accordingly, the unemployment gap is increasingly associated with a smaller output gap. Whilst the Great Recession accelerated that rise, the bulk of the change occurred beforehand.

## 1 Introduction

Okun's law (OL) refers to a reduced-form relationship between cyclical unemployment and output (Okun, 1962). It addresses the issue of how much a country's output is "lost" when unemployment exceeds its natural or trend rate. The relationship provides a link between the labor and goods market over the business cycle, and is often considered a key empirical regularity. It is moreover a core part of many macroeconomic models, where the aggregate supply function is derived from combining OL with the Phillips curve which further links to policy trade offs, Mankiw (2015).

One common and compelling criticism of the Okun's relationship in the literature, though, is the assumption of linearity. Many studies instead suggest that the relationship is characterized by nonlinearities and asymmetries (e.g., Virén, 2001; Cuaresma, 2003; Silvapulle et al., 2004).<sup>1</sup> A nonlinear asymmetric OL would be an important finding. For instance, it would suggest that the effectiveness (and required 'size') of stabilization policy on the real economy would depend in which 'regime' Okun's relationship then lies. Any non linearity in the relationship would also have implications for macroeconomic projections. Moreover, it may affect other recognised economic relationships such as the price and wage Philips curves.

We use a novel methodology to specifically assess the case for asymmetry. Our econometric framework is an augmented version of the logistic smooth transition regression (LSTR) model.<sup>2</sup> The LSTR model nests the linear and standard threshold specification for "low" and "high" values of its identified regimes. It captures smooth transition across regimes, which may be more reasonable in macroeconomics (compared to abrupt transitions), due to various adjustment mechanisms and frictions.

Notably, though, we depart from the bulk of the literature in that we do not

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<sup>1</sup> The stability and usefulness of OL has also been discussed in Knotek (2007) amongst others, and Perman et al. (2015) provides a meta study. Cuaresma (2003) attributes the instability of this relationship to threshold effects: the effect of growth on unemployment being more pronounced in recessions than in expansions.

<sup>2</sup> See van Dijk et al. (2002) for a survey.

consider the threshold variable to be exogenous.<sup>3</sup> Indeed, in as fundamental and deep-seated a relationship as OL, this is intuitive; if there are asymmetries in OL, then it is likely that they should arise from the workings of the economy itself – either from the operation of booms and busts, or from policy shifts or more persistent structural changes. In so far as the threshold is endogenous (i.e., contemporaneously correlated with the disturbance term of the regression), our empirical methodology explicitly seeks to capture that endogeneity. Failure to do so leads to biased and inconsistent estimates. To deal with threshold endogeneity, we draw on the Christopoulos et al. (2019) approach by adjusting the mean of the model conditional on the regimes identified by the threshold variable for the endogenous effects based on copula approach.

Our approach builds up on the work of Kourtellos, Stengos and Tan (2016) (KST), which include bias correction terms conditional on each regime of the model to account for the endogeneity. Our approach, however, enables us to relax the assumption that the threshold variable is normally distributed by relying on copula theory.<sup>4</sup> In addition, we also allow for different variances of the LSTR regression disturbance term and its correlation structure with the threshold variable across the two regimes. We thus control for threshold variable endogeneity effects on the parameter estimates of the LSTR model, by including copula based transformations of the threshold variable truncated at the location parameter value. The marginal distributions of these transformed variables can be estimated based on a non-parametric or density estimation procedure.

This makes our approach of dealing with threshold endogeneity quite general and flexible. Indeed, we avoid the problem of availability of instruments (or ‘weak’ instruments), in the case where one would like to estimate the parameters of the model based on an instrumental variables method (e.g., as first pointed out by Park

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<sup>3</sup> As Kourtellos, Stengos and Tan (2016) discuss, the assumption of threshold exogeneity undermines the practical usefulness of threshold regression models, since many plausible threshold variables (their examples include trade shares, political risk) are very likely to be endogenous to the process under consideration.

<sup>4</sup> See Patton (2006) for a discussion of, and an economic application of, copula methods.

and Gupta, 2012). Furthermore, to test for smooth transition effects in the data, we enhance our approach by suggesting a new likelihood ratio test to detect linearity against LSTR effects under threshold endogeneity. Both the estimation methodology of our approach and the power performance of the suggested *LR* test are evaluated through a Monte Carlo (MC) study.

The paper is organized as follows. [Section 2](#) provides a basic discussion and estimation of OL. [Section 3](#) formally introduces the LSTR model. We demonstrate that, if the threshold variable and disturbance terms in the transition regression are correlated, then estimates of the threshold will be biased (or inconsistent). We then present our approach to adjust the model for the endogeneity of the threshold variables, based on copula theory. The adjusted model is estimated using a two-step concentrated nonlinear least squares method. The method is assessed in a MC study in the appendix. We consider the cases that the threshold variable and disturbance term follow a normal as well as a Student-t distribution. The former allows us to make comparisons to KST's approach (as we have adapted it to the LSTR framework). Generating data from the Student t distribution will show the robustness of the method to fat tails of threshold variables often met in practice.

[Section 4](#) describes and motivates our new linearity testing procedure to test for smooth transition threshold effects against linearity under threshold endogeneity. [Section 5](#) explores threshold choices within the OL framework. We test three broad categories drawn from the literature, classified as (i) *Demand & Cyclical* pressures, (ii) *Structural* features of the economy, and finally, (iii) *Policy & Financial* variables.

Given this background, [Section 6](#) estimates OL assessing the selection of the threshold candidates that indicated nonlinearity in the results of the previous section. These are mostly to be found in the last two categories. We provide estimates based on the standard linear approach, and the LSTR model controlling for endogeneity of the threshold variable based on our suggested approach. We also plot the resulting transition probabilities alongside the threshold as a visual plausibility test. In [Section 7](#), we then combine the most relevant threshold candidates into a composite index.

Section 8 concludes.

## 2 Okun's Law

OL refers to an inverse reduced-form business-cycle relationship between the unemployment rate ( $u$ ) and real output ( $Y$ ). The variables may be in (log or level) first differences (the 'first-differences' model) or expressed in terms of trend-deviations (the 'gap' model). These are respectively, given by,

$$\Delta y_t = \beta_0 + \beta_1 \Delta u_t + \eta_t$$

where  $\Delta y_t = \log(Y_t) - \log(Y_{t-1})$ ,  $\Delta u_t = u_t - u_{t-1}$ , and,

$$y_t^c = \beta_1 u_t^c + \eta_t$$

where  $y_t^c = y_t - y_t^*$  is the deviation of log real GDP from log potential real GDP (i.e., the cyclical component of real GDP:  $y_t^c$ ). Likewise,  $u_t^c = u_t - u_t^*$  is the deviation of the unemployment rate from its 'trend'. Parameter  $\beta_1$  is the Okun coefficient in each case (expected to be negative reflecting the trade off), and  $\eta_t$  denotes a stochastic disturbance. The higher is  $|\beta_1|$ , the *steeper* is Okun's relationship and thus the higher the output costs of a rise in cyclical unemployment.

The gap form requires us to capture latent trends and is related more to understanding business-cycle trade offs. It has the advantage of taking into account the state of the economy relative to its trend or natural rate. The difference form posits a linear relationship between the first difference of the log of output and the first difference of the unemployment rate. The two versions are equivalent if potential growth and the natural rate of unemployment are constant. Since this is unlikely to hold, the gap version appears preferable (and, accordingly, is the one we emphasize).<sup>5</sup>

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<sup>5</sup> Okun's relationship may also be augmented by 'controls': e.g., factor utilization measures, movements in hours worked, capital stocks, the labor force etc – i.e., the 'production function' approach of Pra-



To derive the cyclical component of the series, we explored several different filters common in the literature. In many cases, they gave a relatively similar picture but we chose the Christiano-Fitzgerald one since, for our series of interest, it identified the appropriate frequencies relatively well (i.e., when approximating the ideal band-pass filter over the standard Burns-Mitchell business-cycle frequency of 6 – 32 quarters). **Figure 1** shows both filtered series, the Okun scatter plot, the rolling regression estimates (based on a rolling window of 75 quarters), over 1950q1-2018q4, plus the Andrews (1993), and Andrews and Ploberger (1994)  $F^{\text{SUP}}$  test statistic searching for a break in the above coefficient at unknown date (over the whole sample). Together with the rolling regression estimates, this test statistic can indicate if the Okun relationship is stable and linear relationship, over the sample. If it is not, it can indicate the most likely date after which a break occurs.

Some basic takeaways of asymmetry are revealed by the plots. For example negative output gaps are generally deeper and more abrupt than positive ones (see also Rothman, 1998). The gaps can also be quite changeable: in the first half (or at least second third) of the sample output volatility far exceeded that of unemployment, but became closer in the subsequent decades. Finally, looking at the closeness of the scatter points, we see that the Okun co-movement has been strong at some times and weak at others (suggesting some deviations from linearity). Rolling regression analysis, moreover, suggests that the (unconditional) Okun coefficient has been rising over time. Moreover, that OL may not be stable and linear can also be justified by the  $F^{\text{SUP}}$  test statistic indicating a shift in this coefficient after 2000 (i.e., 2000q4). The 95%

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chowny (1993) (see also Gordon, 1984; Freeman, 2001). We choose to work with the simpler form, since it is the most common representation in the literature (and thus admits easy benchmarking). Moreover, although the appeal of the (production-function) approach is due to its inclusion of these additional margins, many of the proposed controls (in cyclical form) such as hours worked are so heavily correlated with unemployment as to potentially dilute any intended structural interpretation of the parameters. Moreover, the Prachowny derivation is a special case assuming that aggregate production is Cobb Douglas and that the capital stock and a disembodied technology factor are always at their long-run levels; these are counterfactual in the US aggregate data, see Klump et al. (2007).

confidence interval of this date is 1998q1:2006q1.<sup>6</sup>

Indeed, over our sample the economy has undergone several major events: the productivity slowdown of the early 1970s, the major oil shocks in that decade, and the “Great Moderation” of reduced macroeconomic volatility from the mid 1980s onwards (variously attributed to ‘better’ monetary-policy stabilization, structural change, greater labor flexibility). Regarding the latter, a period of prolonged expansion may have affected the Okun trade off by enticing more marginal, potentially more flexible workers into the labor force. Moreover, such enhanced economic predictability and perceived economic prospects may have encouraged more financial leverage, as reflected in the build up of public and private debt. Finally, there was also the financial crisis from the late 2000s onwards (the “Great Recession”), followed in turn by extraordinary monetary (and to a lesser extent fiscal) policy accommodation.

There were also, though, major shifts in cyclical patterns from the mid 1980s onwards, see Fernald and Wang (2016): labor productivity turned from pro to counter cyclical – largely reflecting the weakening pro cyclical of factor utilization and hence of raw TFP growth.<sup>7</sup> Weakening pro-cyclical in utilization points to reduced factor hoarding. This may be due to increased flexibility in the economy, as reflected by the expansion of female labor participation and perhaps declining labor power; it may also reflect the decline of manufacturing and manufacturing employment (where utilization was traditionally a more important margin of adjustment).

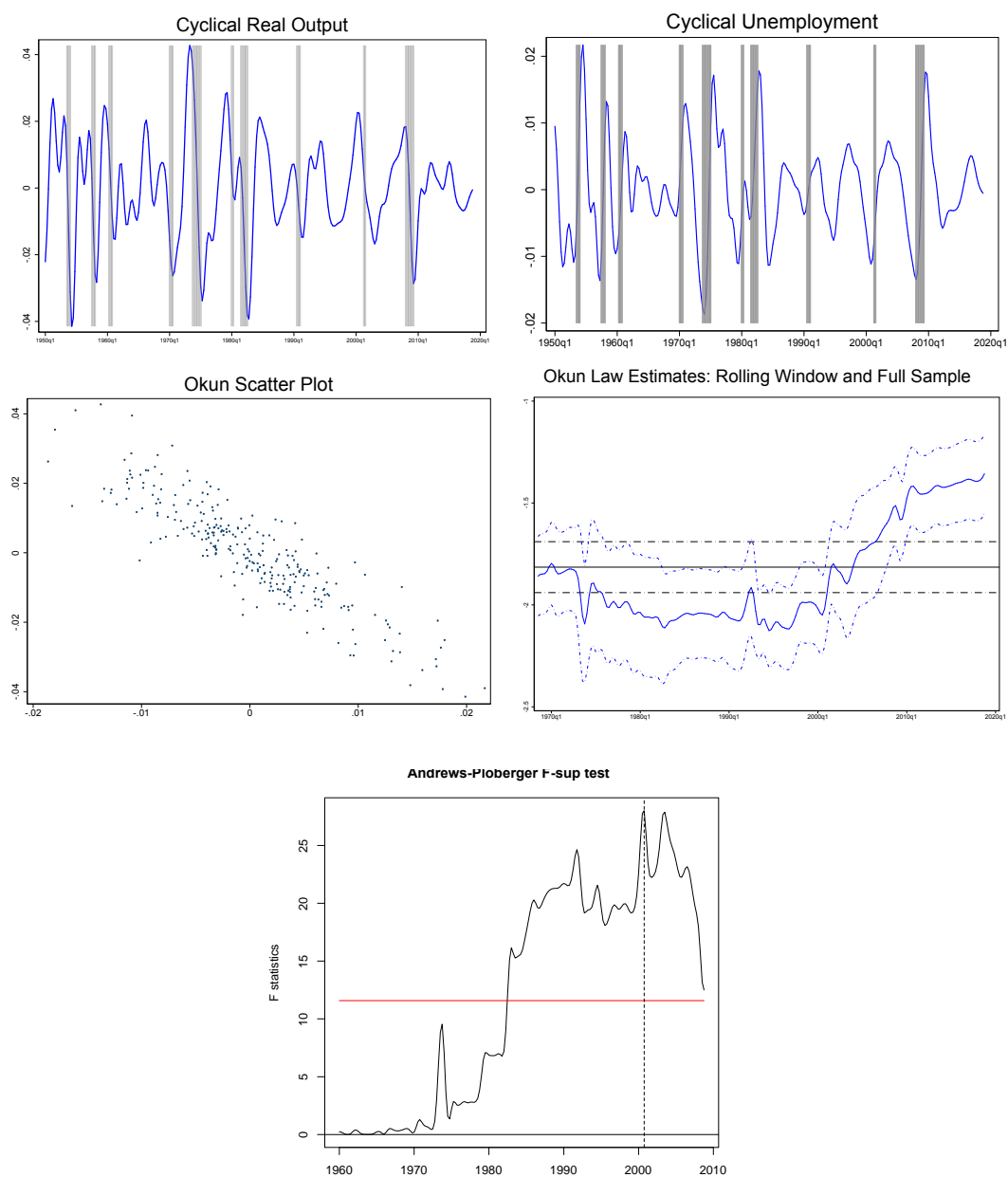
Estimates of OL (both variants) are shown in [Table 1](#). Results corroborate those of the simple scatter plot. They suggest that a unit increase of cyclical unemployment is associated with a decline of just under 2% in output (in cyclical or difference terms). Indeed in line with Okun’s original work, most US studies locate the (absolute) coef-

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<sup>6</sup> The critical values of the  $F^{\text{sup}}$  test statistic are based on Hansen’s (1997) algorithm.

<sup>7</sup> To see why, we can decompose labor productivity growth into growth in TFP, capital deepening and labor quality. Since the latter two are robustly counter-cyclical, the change in labor productivity essentially comes from the weakening pro cyclical of raw TFP growth. Indeed in recent decades TFP growth (net of its ‘pure’ component) has become so much less pro-cyclical that overall labor productivity is now judged net counter cyclical.

FIGURE 1: Okun Law Variables, Correlations and Stability



Notes: Shaded gray areas represents NBER recession dates at quarterly frequency. In the rolling window coefficients graphs (for the gap version of the Okun's Law) the 95% standard errors are given by the blue dashed lines (the window size is 75 quarters).

TABLE 1: Okun Law Coefficients

Parameters	$\Delta y_t$ †	$y_t^c$ ‡
$\beta_0$	0.007 (0.004)	
$\beta_1$	-1.661 (0.107)	-1.815 (0.068)
<i>AIC</i>	-993.318	-986.366

Notes:

† First-differences form:  $\Delta y_t = \beta_0 + \beta_1 \Delta u_t + \eta_t$  where  $\Delta y_t = \log(Y_t) - \log(Y_{t-1})$ , and  $\Delta u_t = u_t - u_{t-1}$ . Calculations based on year-on-year changes generate similar results. The raw data are taken from Federal Reserve Bank of St. Louis (FRED): the unemployment Rate (*UNRATE*) and Real GDP (*GDPC1*).

‡ The gap form is given by,  $y_t^c = \beta_1 u_t^c + \eta_t$  where  $y_t^c = y_t - y_t^*$  is the deviation of log real GDP from log potential real GDP (i.e., the cyclical component of real GDP:  $y_t^c$ ). Likewise,  $u_t^c = u_t - u_t^*$  is the deviation of the unemployment rate from its 'trend'.

Numbers in brackets below the coefficients represent bootstrapped standard errors. *AIC* denotes the Akaike Information criterion (for a given model size, the lowest AIC score is preferred).

ficient in a (1, 3] interval. That the coefficient typically exceeds one in absolute value (i.e., cyclical output drops *by more* than the increase in cyclical unemployment) reflects different adjustment margins that amplify movements in unemployment on output. For example some fraction of the unemployed may cease job search thus reducing the labor force, labor productivity may fall (reflecting labor hoarding), hours worked may fall, and the economy may weaken through the normal Keynesian spending multiplier.

### 3 A LSTR Model with Threshold Endogeneity

A type of structural threshold model that we can employ to investigate threshold effects is the following two-regime logistic smooth transition regression (LSTR) model:<sup>8</sup>

$$y_t = x_t' \beta_1 (1 - \mathbf{g}(z_t; \gamma, \delta)) + x_t' \beta_2 \mathbf{g}(z_t; \gamma, \delta) + \varepsilon_t, \quad (1)$$

where  $y_t = y_t^c$  and  $x_t = (1, u_t^c)'$  is a  $(M \times 1)$  vector of independent variables, with  $M = 2$ ,  $\beta_h = (\beta_1, \beta_2)'$  denotes the vector of the slope coefficients of the elements of vector  $x_t$  in two distinct regimes, where  $h = \{1, 2\}$  denotes the two regimes “1” and “2”, and  $\varepsilon_t \sim IID(0, \sigma_\varepsilon^2)$  is the disturbance term. Function  $\mathbf{g}(z_t; \gamma, \delta)$  is a continuous logistic function of the observable variable  $z_t$ , known as the threshold variable, which governs the transition between the two regimes:

$$\mathbf{g}(z_t; \gamma, \delta) = \frac{1}{1 + \exp(-\gamma(z_t - \delta))}. \quad (2)$$

The value of  $\delta \in \mathbb{R}$  is known as the location or threshold parameter, which defines the two regimes. Parameter  $\gamma > 0$ , the speed-of-transition parameter, determines the smoothness of the transition from one regime to the other. When  $\gamma \rightarrow \infty$ ,  $\mathbf{g}(z_t; \gamma, \delta)$  tends to indicator function  $\mathcal{I}(z_t > \delta)$ , for all  $i$ . In this case, the LSTR model can be approximated by the standard threshold model and thus, the transmission between regimes is abrupt: the shift from regime “1” to “2” becomes instantaneous at  $z_t = \delta$ . On the other hand, when  $\gamma \rightarrow 0$ , then  $\mathbf{g}(z_t; \gamma, \delta) \rightarrow \frac{1}{2}$ . In this case, the LSTR model reduces to a linear model, with vector of parameters  $\beta = \frac{1}{2}(\beta_1 + \beta_2)$ .

Through transition function  $\mathbf{g}(z_t; \gamma, \delta)$ , the slope coefficients of model (1) are time-varying, depending on the value of  $z_t$ . One econometric problem often encountered in practice, though, when estimating model (1) is the endogeneity of the threshold

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<sup>8</sup> See Teräsvirta (1994). Note that the inclusion of a (potentially regime-specific) intercept in the LSTR model algebra is for generality.

variable  $z_t$  with the disturbance term  $\varepsilon_t$ . Let us therefore split the sample of  $z_t$  across the two regimes as follows:  $\mathcal{Z}_1 = (-\infty, \delta]$  and  $\mathcal{Z}_2 = (\delta, \infty)$ , based on a value of location parameter  $\delta$ , and assume that the disturbance term  $\varepsilon_t$  is distributed differently across the two regimes, i.e.,  $\varepsilon_{h,t} \sim IID(0, \sigma_h^2)$ ,  $h = \{1, 2\}$ . Endogeneity between  $z_t$  and  $\varepsilon_{h,t}$  means that  $\mathbb{E}(\varepsilon_{1,t}|\mathcal{Z}_1) \neq 0$  and  $\mathbb{E}(\varepsilon_{2,t}|\mathcal{Z}_2) \neq 0$ , which implies that estimates of the location parameter  $\delta$  will be biased (or inconsistent). This in turn implies that the estimates of all slope coefficients of model (1) will also be biased.

There are different estimation methods to tackle this endogeneity problem in econometrics (Antonakis et al. (2014) offer a survey). The method chosen here draws on that of Christopoulos et al. (2018) which employs copulas to capture the dependence between the disturbance term  $\varepsilon_t$  and the threshold variable  $z_t$ .<sup>9</sup> This method allows for the level of dependence between  $\varepsilon_{h,t}$  and  $z_t$ , as well as the variance of  $\varepsilon_{h,t}$  to be different across the two regimes. Copulas are functions which can express joint probability distributions (or densities) of random variables  $\varepsilon_{h,t}$  and  $z_t$  in terms of their marginal probability density functions (PDFs) and a copula function capturing the dependence between  $\varepsilon_{h,t}$  and  $z_t$ . Based on the copula function, we can derive a single-correlation structure of  $\varepsilon_{h,t}$ ,  $h = 1, 2$ , which can be used to control the endogeneity of  $z_t$  in model (1). To this end, consider the following definition:

**Definition:** Let  $p_\delta \in (0, 1) = P(z_t \leq \delta) = F_z(\delta)$ , where  $F_z$  is the distribution function of  $z_t$ . The joint distribution of the pair of random variables  $(\varepsilon_{h,t}, z_t \in \mathcal{Z}_h)$ ,  $h = \{1, 2\}$ , can be written based on copulas as follows:

$$F_{\varepsilon_h z}(\varepsilon_{h,t}, z_t | \mathcal{Z}_h) = C_h \left( F_{\varepsilon_h}(\varepsilon_{h,t}), F_{z|\mathcal{Z}_h}(z_t | \mathcal{Z}_h) \right),$$

where  $C_h$  is a bivariate appropriately scaled copula, with  $C_h : [0, 1]^2 \rightarrow [0, 1]$ ,  $F_{\varepsilon_h}$  is

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<sup>9</sup> Although in their study they considered the simpler TAR model.

the marginal distribution function of  $\varepsilon_{h,t}$ , and

$$F_{z|\mathcal{Z}_1}(z_t|\mathcal{Z}_1) = \frac{F_z(z_t)}{p_\delta} \quad \text{if } 0 \leq F_z(z) \leq p_\delta$$

$$F_{z|\mathcal{Z}_2}(z_t|\mathcal{Z}_2) = \frac{F_z(z_t) - p_\delta}{1 - p_\delta} \quad \text{if } p_\delta \leq F_z(z) \leq 1,$$

are the truncated from above and below location parameter value  $\delta$  distribution functions of  $z_t$ , respectively.

From the above definition, it is clear that the truncated joint distribution of the pair of variables  $(\varepsilon_{h,t}, z_t)$ ,  $F_{\varepsilon_h z}(\varepsilon_{h,t}, z_t|\mathcal{Z}_h)$ , for  $h = \{1, 2\}$ , constitutes a copula  $C_h$  on  $[0, 1]^2$  with uniformly distributed on  $[0, 1]$  margins, since  $F_{z|\mathcal{Z}_h}(z_t|\mathcal{Z}_h)$  is appropriately scaled to integrate to unity. Based on copula theory, the conditional distribution function of  $\varepsilon_{h,t}$  on  $z_t \in \mathcal{Z}_h$  can be derived from  $C_h$  as follows:

$$F_{\varepsilon_h|z}(\varepsilon_{h,t}|\mathcal{Z}_h) = \frac{\partial}{\partial F_{z|\mathcal{Z}_h}} C_h \left( F_{\varepsilon_h}(\varepsilon_{h,t}), F_{z|\mathcal{Z}_h}(z_t|\mathcal{Z}_h) \right), \quad (3)$$

while the conditional probability density function related to this distribution is given as,

$$\begin{aligned} f_{\varepsilon_h|z}(\varepsilon_{h,t}|\mathcal{Z}_h) &= \frac{\partial^2}{\partial \varepsilon_h \partial F_{z|\mathcal{Z}_h}} C_h \left( F_{\varepsilon_h}(\varepsilon_t), F_{z|\mathcal{Z}_h}(z_t|\mathcal{Z}_h) \right) \\ &= c_h \left( F_{\varepsilon_h}(\varepsilon_t), F_{z|\mathcal{Z}_h}(z_t|\mathcal{Z}_h) \right) f_{\varepsilon_h}(\varepsilon_{h,t}), \end{aligned} \quad (4)$$

where  $c_h \left( F_{\varepsilon_h}(\varepsilon_t), F_{z|\mathcal{Z}_h}(z_t|\mathcal{Z}_h) \right) = \frac{\partial^2}{\partial F_{\varepsilon_h} \partial F_{z|\mathcal{Z}_h}} C_h \left( F_{\varepsilon_h}(\varepsilon_t), F_{z|\mathcal{Z}_h}(z_t|\mathcal{Z}_h) \right)$  is the copula density function corresponding to  $C_h$  and  $f_{\varepsilon_h}(\varepsilon_t) = \frac{\partial}{\partial \varepsilon_h} F_{\varepsilon_h}(\varepsilon_t)$  is the probability density of  $\varepsilon_{h,t}$ .<sup>10</sup> Based on the above relationships for  $f_{\varepsilon_h|z}(\varepsilon_{h,t}|\mathcal{Z}_h)$  and  $F_{\varepsilon_h|z}(\varepsilon_{h,t}|\mathcal{Z}_h)$ ,

<sup>10</sup> Note that the joint probability density corresponding to  $F_{\varepsilon z}(\varepsilon_t, z_t|\mathcal{Z}_h)$  is given as follows:

$$f_{\varepsilon z}(\varepsilon_t, z_t|\mathcal{Z}_h) = c_h \left( F_{\varepsilon}(\varepsilon_t), F_{z|\mathcal{Z}_h}(z_t|\mathcal{Z}_h) \right) f_{\varepsilon}(\varepsilon_t) f_{z|\mathcal{Z}_h}(z_t|\mathcal{Z}_h)$$

it can be shown that, in the case where  $C_h$  is Gaussian and  $\varepsilon_{h,t} \sim \mathcal{N}(0, \sigma_h^2)$ , which is often assumed in regression models,  $\varepsilon_{h,t}$  has the following single-factor correlation structure:<sup>11</sup>

$$\varepsilon_{h,t} = \omega_h z_{h,t}^* + \text{Var}(\varepsilon_{h,t}|z_{h,t}^*)^{1/2} u_{h,t}, \quad h = \{1, 2\} \quad (5)$$

where  $z_{h,t}^*$  constitutes a transformation of  $z_t$  which is distributed as  $\mathcal{N}(0, 1)$  and defined as,

$$z_{h,t}^* = \Phi^{-1} \left( F_{z|\mathcal{Z}_h}(z_t|\mathcal{Z}_h) \right)$$

where  $\Phi^{-1}$  is the quantile function of the standard normal distribution. Moreover,

$$\omega_h = \sigma_h \rho_{u_h, z_h^*}$$

where  $\rho_{u_h, z_h^*}$  is the Pearson correlation coefficient between  $\varepsilon_{h,t}$  and  $z_{h,t}^*$ ,  $u_{h,t}$  is an  $IID(0, 1)$  disturbance is independent of  $z_t^*$  and

$$\text{Var}(\varepsilon_{h,t}|z_{h,t}^*) = \sigma_h^2(1 - \rho_{u_h, z_h^*}^2).$$

Using the above representation of relationship of  $\varepsilon_{h,t}$  enables us to write model (1) as,

$$y_t = \begin{cases} x_t' \beta_1 (1 - \mathbf{g}(z_t; \gamma, \delta)) + \omega_1 z_{1,t}^* + e_{1,t} & \text{if } z_t \in \mathcal{Z}_1 : \text{Regime "1"} \\ x_t' \beta_2 \mathbf{g}(z_t; \gamma, \delta) + \omega_2 z_{2,t}^* + e_{2,t} & \text{if } z_t \in \mathcal{Z}_2 : \text{Regime "2"} \end{cases} \quad (6)$$

where  $e_{h,t} = -\omega_h z_{h,t}^* + \varepsilon_{h,t}$  is a disturbance term with conditional mean  $\mathbb{E}(e_t|z_{h,t}^*) = 0$  and conditional variance  $\text{Var}(e_{h,t}|z_{h,t}^*) = \sigma_\varepsilon^2(1 - \rho^2)$ , since  $\mathbb{E}(\varepsilon_{h,t}|z_{h,t}^*) = \omega_h z_{h,t}^*$  from

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where  $f_{z|\mathcal{Z}_1}(z_t|\mathcal{Z}_1) = \frac{f_z(z_t)}{p_\delta}$ ,  $-\infty < z_t \leq \delta$ , and  $f_{z|\mathcal{Z}_2}(z_t|\mathcal{Z}_2) = \frac{f_z(z_t)}{1-p_\delta}$ , with  $\delta < z_t \leq \infty$ , are the truncated from above and below  $p_\delta$  probability densities  $f_z(z_t)$ .

<sup>11</sup> See, for example, Joe (2014), Christopoulos et al. (2018).



(5). The last, augmented with random variables  $z_{h,t}^*$  version of model (1) can be employed to control for the endogeneity problem of  $z_t$ . The random variables  $z_{h,t}^*$  added to the rhs of (6), correct the conditional mean of  $y_t$  on  $Z_h$ , for  $h = \{1, 2\}$ , for the contemporaneous correlation between  $\varepsilon_{h,t}$  and  $z_t$  (implying  $\mathbb{E}(\varepsilon_{h,t}|Z_h) \neq 0$ ). This is done, note, without making any assumption about the distribution of  $z_t$ . Furthermore, as already highlighted, it allows the distribution of  $\varepsilon_{h,t}$  and its correlation structure with  $z_t$  to change across the two regimes.

### 3.1 Estimation Aspects and Monte Carlo Results

Model (6) can be employed to estimate the location parameter  $\delta$  and its remaining parameters collected in vector  $\theta(\delta) = (\gamma, \beta_h, \omega_1, \omega_2)'$  based on a two-step nonlinear least squares (NLLS) method, since the disturbance term  $u_{h,t}$  and, hence,  $e_{h,t}$  is independent of the transformed variable  $z_{h,t}^*$  for  $h = \{1, 2\}$ .

In particular,  $\delta$  can be estimated, in a first step, by solving the following NLLS optimization problem:

$$\hat{\delta} = \arg \min_{\delta \in Q_z} RSS(\delta),$$

where  $RSS(\delta) = \sum_{t=1}^T \hat{e}_{h,t}^2$  is the residual sum of squares of (6),  $\delta$  is an interior point of  $Q_z$ , since we assume  $p_\delta = P(z_t \leq \delta) \in (0, 1)$ . To estimate  $\delta$ , note that we require values of the transformed variables  $z_{h,t}^*$  given by  $\Phi^{-1} \left( F_{z|Z_h}(z_t|Z_h) \right)$ . This can be done based on non-parametric estimates of the marginal distribution  $F_{z|Z_h}(z_t|Z_h)$  (see Silverman, 1986), or based on the empirical cumulative distribution function, denoted ECDF. Given the optimal estimate of  $\hat{\delta}$ , the slope parameters of model (1) and the speed-of-transition coefficient  $\gamma$  collected in vector  $\theta(\hat{\delta})$  can be estimated, in a second step. Following the literature on threshold models (see, e.g., Chan, 1993; Samia and Chan, 2011), the estimator  $\hat{z}_\delta$  is  $T$ -consistent and the estimates of vector  $\theta(\hat{\delta})$ , which correspond to  $\hat{\delta}$ , are  $\sqrt{T}$  asymptotically normal.

As a remark on the above estimation procedure note that, instead of the one dimensional grid search over  $\delta$ , we can carry out a two dimensional grid search over  $\delta$

and  $\gamma$  (see, e.g., Leybourn et al., 1986; and Franses and van Dijk, 2000). This procedure may mitigate optimization problems in estimating  $\gamma$ , due to the nonlinear nature of function  $g(z_t; \gamma, \delta)$ . Furthermore, it can have better small sample properties, due to grid-search process in estimating  $\delta$  and  $\gamma$ . Given the estimates of  $\delta$  and  $\gamma$ , then we can estimate the remaining slope parameters, collected in vector  $\beta_h$ , in a second step.

To evaluate the performance of our estimation approach to successfully control for the endogeneity of threshold variable, we carry out a small MC study, see [Appendix A](#). We consider cases in which the disturbance term  $\varepsilon_{h,t}$  and threshold variable  $z_t$  are jointly normally and then Student-t distributed, with different values of the speed-of-transition parameter  $\gamma$  and sample sizes  $T$ . Generating data from the Student's t distribution, which allows for tail dependence between  $u_{h,t}$  and  $z_t$ , can show the robustness of our method to such features in the data.

The results of our MC clearly supports the view that our method can successfully control for the endogeneity of  $z_t$  on the estimates of location parameter  $\delta$ . They also show that ignoring this endogeneity leads to series biases in the estimates of  $\delta$ . These results hold for both the distributions of  $\varepsilon_{h,t}$  and  $z_t$  considered, meaning that our method is robust to misspecification of these distributions. For the case that  $\varepsilon_{h,t}$  and  $z_t$  are normally distributed, our method compares favorably to that of Kourtellos et al. (2016) based on the inverse Mills ratios, modified appropriately for the LSTR model (1). Also, our method is robust to the case that both  $\varepsilon_{h,t}$  and  $z_t$  follow the Student-t distributed. Finally, another interesting result of our MC exercise is that our method can be implemented without any concern for bias or inefficiency of the estimates of  $\delta$  in the case where the threshold variable  $z_t$  is exogenous.

## 4 A New Test for Linearity Controlling for Threshold Endogeneity

Before estimating model (1), or its extended version (6), a critical prior testing procedure is to diagnose if the data supports our threshold model compared to its linear

specification. To this end, we suggest a suitable testing procedure.

We follow recent work in the literature on threshold or LSTR models (see, e.g., Hansen (1996) and KılıÇ, 2016) which is focused on testing  $H_0: \beta_1 = \beta_2$  (implying  $\gamma \rightarrow 0$ ) against  $H_a: \beta_1 \neq \beta_2$  (implying  $\gamma > 0$ ). As noted by KılıÇ (ibid), compared to inference procedures testing for the exclusion restrictions on the threshold variable  $z_t$  or its product terms with regressors collected in vector  $x_t$  based on an approximation of model (1) under  $H_0$ , our suggested procedure may be proved more powerful for  $\gamma$  values far away from the  $\gamma = 0$  neighborhood.<sup>12</sup> That is to say where the approximation of (1) is not accurate and depends on the value of the location parameter  $\delta$ . Furthermore, in our simulations we found that endogeneity of threshold variable  $z_t$  makes the power performance of the inference procedures based on the above approximation of (1) even worse, due the estimation bias of the slope coefficients of the auxiliary regression.

More specifically, to carry out a test of  $H_0: \beta_1 = \beta_2$  against  $H_a: \beta_1 \neq \beta_2$ , we rely on the likelihood ratio ( $LR$ ) test statistic

$$LR(\gamma, \delta) = 2(\log L(\theta(\delta)) - \log L(\beta))$$

where  $\log L(\theta(\delta))$  and  $\log L(\beta)$  constitute the maximum log-likelihood function of model (1) under the alternative and null hypotheses, respectively. Note that, in order to estimate the model under the alternative hypothesis, we will use the auxiliary regression (6), controlling for the endogeneity of the threshold variable  $z_t$ . Since the nuisance parameters  $\delta$  and  $\gamma$  are not identified under the null, we next suggest a sup-version of statistic  $LR(\gamma, \delta)$  (see Andrews and Ploberger, 1994), defined as follows:

$$LR^{\text{sup}} \equiv \sup_{(\gamma, \delta) \in Q_\gamma \times Q_\delta} LR(\gamma, \delta),$$

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<sup>12</sup> The initial literature of tests for linearity against threshold specifications (such as Luukonen et al., 1998) replaced the transition function by a Taylor series expansion which is estimable under the null: with the test amounting to testing for the significance of the interaction of the linear regressors with the polynomial terms. See also Escribano and Jordà (1999), van Dijk et al. (2002).

where  $Q_\gamma$  denotes a compact subspace on the real line, searching for an optimal value of  $\gamma$  and  $Q_z$  is defined as before.<sup>13</sup> Since the distribution of  $LR^{\text{sup}}$  is non standard under the null, its critical values can be obtained based on a parametric bootstrap procedure, generating data under the null hypothesis of linearity (i.e.,  $H_0: \beta_1 = \beta_2$ ).

To evaluate the power performance of statistic  $LR^{\text{sup}}$ , we carry out a MC exercise in [Appendix B](#). In this exercise, we also consider the case that we ignore the endogeneity of threshold variable  $z_t$ . The results of this exercise indicate that  $LR^{\text{sup}}$  has satisfactory power. This is true under alternative copula functions and marginal distributions of  $\varepsilon_t$  and  $z_t$  considered in the MC exercise. Another interesting finding is that our test does not lose significant power in the case where  $z_t$  is exogenous.

## 5 Data and Threshold Candidates

Which variables might be associated with threshold effects in OL?

So far we have discussed the possibility of (endogenous) threshold variables leading to threshold effects in Okun's law, without discussing which would be those variable(s). We take an agnostic approach; rather than imposing a particular favored threshold variable, we explore a number of candidates reflecting discussions in the literature. A common feature to all, consistent with our methodology, is their likely endogeneity to the Okun trade off.

Indeed, as already noted, OL is a reduced form relationship, its strength reflects several influences. In that vein, Fernald et al. (2017) and Daly et al. (2018) decompose Okun's coefficient emphasizing the different margins and adjustment channels that firms and households use to respond to different shocks.<sup>14</sup> Although these authors do not explore nonlinearity, their framework is nonetheless suggestive. For instance, labor hoarding (an empirically well-established phenomenon) cushions the unemploy-

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<sup>13</sup> Note that, in practice,  $Q_\gamma$  can be set to  $Q_\gamma = \left[ \frac{1}{10\sigma_z}, \frac{1000}{\sigma_z} \right]$ , where  $\sigma_z$  is the standard deviation of the threshold variable  $z_t$ , see KılıÇ (2011).

<sup>14</sup> To illustrate, respectively, these are the extensive (e.g., labor force participation and migration etc.) and intensive margins (e.g., hours per worker, factor utilization etc.)

ment consequences of downturns perceived to be temporary or “small”. However, a sufficiently large downturn (i.e., in our context, beyond some estimated threshold) may counter labor hoarding incentives, thus changing the output-unemployment nexus. To take another example, a deteriorating sovereign debt outlook may, beyond some point, crowd out activity, impair financial markets and heighten uncertainty, with implications for labor participation and factor intensity.<sup>15</sup> That said, there may be *many* feasible threshold candidates rather than merely a sparse outcome, in effect candidates which may reinforce or counteract one another’s effects.<sup>16</sup> Accordingly, we also examine the performance of a composite indicator.

We classify these in three broad (and not necessarily mutually exclusive) categories: (i) *Demand & Cyclical pressures*; (ii) *Structural*; and (iii) *Policy & Financial* (see [Table 2](#)).<sup>17</sup> The main data source is the Federal Reserve Bank of St. Louis (FRED) and the BLS (listed are symbol names, plus, where relevant their cyclical equivalent). As before, the sample spans 1950q1 to 2018q4. In the final column, we show the corresponding  $LR^{\text{sup}}$  test statistic and probability values for the null of linearity against the LSTR specification (performed for the Gaussian copula).

Consider the first category of threshold candidates, *Demand & Cyclical*. Many studies suggest Okun’s coefficient moves over the phases of the business cycle, and in a nonlinear, asymmetric fashion (e.g., Lee, 2000; Harris and Silverstone, 2001; Mayes and Virén, 2001; Cuaresma, 2003; Silvapulle et al. 2004). Common rationales for this cyclical asymmetry include: labor hoarding; downward wage rigidity; employment regulations; cyclical firm growth; non-constant factor substitution (Courtney, 1991) etc. In line with this business-cycle asymmetry literature, we consider as possible threshold variables the unemployment and ‘natural’ unemployment rate, the output

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<sup>15</sup> Reinhart et al. (2012) discuss threshold behaviour in the growth-debt nexus.

<sup>16</sup> For an analysis of sparse and dense models, see Giannone et al. (2019).

<sup>17</sup> Many of these are quite representative of macro US series commonly used for econometric studies, and most are included, for instance, in the large dat set of Marcellino et al. (2006) (their economy-wide variables anyway).

gap, output, and inflation rates (which similarly reflect cyclical demand pressures).<sup>18</sup>

Our second category is dubbed *Structural*. Again this reflects various strands of the Okun literature as well as more general themes of structural change in the post-war US economy: namely, that various secular (though not necessarily irreversible) trends may have imparted an effect on Okun's coefficient. For example female labor-market participation has risen, the employment share of manufacturing has declined as has the labor income share, the economy experienced a 'Great Moderation' of reduced volatility etc. It is not unreasonable to suppose that such changes affect the Okun trade off.

The final category assesses *Policy & Financial* variables as threshold candidates in the Okun specification. We include a variety of interest rates,<sup>19</sup> private and public debt ratios. Policy and financial measures, by affecting the different margins underlying Okun's relationship, may thus have an influence on the Okun Law parameters.

Table 2 shows around half the threshold candidates reject linearity (often at or below 1%). Among the first category, demand or cyclical factors, only inflation and capacity utilization come to close to significance (at 10%). This finding is interesting since these have often been the favored candidates of past Okun threshold studies (e.g., Virén, 2001). This, however, need not imply that cyclical pressures do not impact Okun law parameters. But it does suggest that downturns may have long

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<sup>18</sup> Note that dichotomous (Heaviside) threshold variables (of the expansion/contraction variety) which are often used in such analyses, such as

$$z_t = \begin{cases} 1 : \text{Expansion} & \text{if } u^c < 0, \text{ or } y^c > 0 \\ 0 : \text{Contraction} & \text{otherwise} \end{cases}$$

are not feasible threshold variable types in our framework since they are not continuous.

<sup>19</sup> We also considered the Corporate Bond Credit Spread of Gilchrist and Zakrajsek (2012), and the National Financial Conditions Index (from the FRB Chicago). But their short sample (available from 1973 and 1971, respectively) precluded their use. Although the credit spreads share similar information with the interest rates used since, for instance, they reflect similar monetary or fiscal shocks. Likewise, many variables which might have been of interest, for instance inequality measures, tax burden, 'economic and policy uncertainty' metrics etc were excluded since they tend to be of short sample and/or not available at a quarterly frequency. Further details of these data choices and restrictions are available on request.

and persistent effects; they may induce protracted adjustments in Okun margins that extend well beyond the defined periods of recession.

The second (structural) category, also has some valid threshold candidates. The rolling volatility measures, indicative of the Great Moderation and perhaps reduced macroeconomic risk, can reject linearity. The labor share<sup>20</sup>, manufacturing employment share, long term unemployed (percentage unemployed for  $\geq 27$  weeks), and female participation rates, illustrating important shifts in labor-market features, are also detected.

Finally, policy and financial measures, yield a number of valid threshold candidates. Both long and short policy rates, the shadow short rate (Wu and Xia, 2016) as well as debt ratios, reject the linear specification and are thus viable threshold candidates.

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<sup>20</sup> Note, the variant of the labor share that we use is the quarterly indexed one provided by FRED where 2012 = 100. For comparison purposes we annualized this series and compared it to the Share of Labor Compensation series provided by University of Groningen (also available from FRED), see supplementary material [Figure C.1](#).

TABLE 2: Data: Definitions, Sources and Linearity Testing

Data Series	Symbol Source (mnemonic)	$LR^{\text{sup}}$
<b>Demand</b>		
Cyclical Unemployment Rate	$u^c$ FRED (UNRATE)	2.082 [0.266]
Cyclical Real GDP	$y^c$ FRED (GDPFC1)	6.145 [0.189]
Output Gap (% of potential GDP)	$og$ CBO	5.691 [0.360]
Capacity Utilization	$cu$ FRED (TCU)	8.723 [0.060]
Consumer Price Inflation (% $\Delta$ yoy)	$\pi$ FRED (CPIAUCSL)	6.422 [0.058]
Commercial Real Estate Price Inflation	$\pi_H$ FRED/Haver Analytics	6.145 [0.180]
<b>Structural</b>		
Rolling 10-year St. Dev. of Cyclical Output (Risk)	$\sigma_{y_t^c(10)}$	16.490 [0.000]
Utilization-Adjusted TFP Growth	$gTFP$ Fernald (2018)	5.989 [0.141]
Non-farm Business Sector: Labor Share	$labsh$ FRED (PR85006173)	39.669 [0.000]
Employment Share: Manufacturing	$lmanu$ FRED (MANEMP/PAYEMS)	24.739 [0.000]
Labor Force Participation Rate (Total; Female)	$pr; pr_f$ FRED (CIVPART; LNS11300002)	5.867 [0.112]; 19.036 [0.001]
Natural Rate of Unemployment	$u_n$ CBO	19.518 [0.000]
Long-Term Unemployed*	$u_l$ Bureau of Labor Statistics	31.459 [0.000]
<b>Policy &amp; Financial</b>		
Federal Funds Rate; Long Term Interest Rates (10y Gov. Bonds)**	$i_s; i_l$ FRED (FEDFUNDS; IRLTLT01USM156N)	30.420 [0.000]; 29.029 [0.000]
Shadow Policy Rate**	$i_{s,wx}$ Wu and Xia (2016)	31.184 [0.000]
Corporate Bond Spread***	$i_{Baa}$ FRED (BAAFFM)	4.188 [0.166]
Total Debt to GDP (Private NFC; level, cycle)	$dp; dp^c$ FRED and Haver Analytics	16.664 [0.000]; 1.114 [0.421]
Total Debt to GDP (Federal Government; level, cycle)	$d; d^c$ FRED/Haver Analytics	19.472 [0.000]; 1.746 [0.316]

Notes: All series are quarterly and, where relevant, seasonally adjusted and span 1950q1-2018q4 except  $i_s, i_{s,wx}$  which start in 1954q3,  $i_l$  (1960q1),  $d_p$  (1951q4),  $cu$  (1967q1).

\* Unemployed for  $\geq 27$  weeks (% of civilians unemployed).

\*\* Calculated as end of period. \*\*\* Moody's Baa Corporate Bond minus Federal Funds Rate

Numbers in brackets in the final column are bootstrapped probability-values of the threshold linearity tests.



## 6 Estimation of the LSTR Model for Okun's Law

Having presented our empirical methodology to estimate a LSTR model under endogeneity of the threshold variable, and pretested for linearity, we now estimate the model. To measure directly the magnitude of the shift in the slope coefficient of unemployment, we estimate an augmented version of (1) with the regressors  $z_1^*$  and  $z_2^*$ :

$$y_t^c = \beta_1 u_t^c + \beta_2 u_t^c \left[1 + e^{-\gamma(z_t - \delta)}\right]^{-1} + \omega_1 z_1^* (\mathcal{I}(z_t \in \mathcal{Z}_1)) + \omega_2 z_2^* (\mathcal{I}(z_t \in \mathcal{Z}_2)) + e_t. \quad (7)$$

This version nests the no-threshold case,  $\beta_2 = 0$ , wherein the Okun's coefficient is given solely by  $\beta_1$ . Otherwise, it is given by  $\beta_1$  in the first regime and  $\beta = \beta_1 + \beta_2$  in the second. Thus,  $\beta_2$  captures the magnitude of the shift of the slope coefficient of unemployment from the first regime to the second, relative to the first.

Table 3 presents model estimates for two cases: Panel A ignores endogeneity of the threshold variable and Panel B allows for it. The latter is based on our preferred estimation approach already presented, including the copula-transformed variables  $z_{1,t}^*$  and  $z_{2,t}^*$  into the rhs of (7) with the attendant parameters  $\omega_1$  and  $\omega_2$ . For better small sample estimation properties, we present bootstrapped standard errors (in parentheses). These are calculated based on a wild parametric bootstrap method (see Davidson and MacKinnon, 2007). For  $\gamma$  and  $\delta$ , confidence intervals are shown in braces, based on that bootstrap method. We also present values of information criteria in order to compare the model with its earlier linear (no-regime) specification.<sup>21</sup>

### 6.1 Results

Table 3 shows the estimated  $\beta$  Okun parameters (again mostly within the 1, 3 range), and the threshold and speed of adjustment parameters,  $\delta$  and  $\gamma$  respectively. In the endogenous case, we also show the  $\omega$  parameters (we scale up by 100 for legibility).

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<sup>21</sup> Supplementary Table C.1 presents summary statistics of the threshold variables.

Finally we present the *AIC* and the sample percentage residing in the second regime, denoted  $\mathcal{D}_2$ .

To aid interpretation consider the case,

$$\beta_1 < \beta < 0, \tag{8}$$

where, to recall  $\beta = \beta_1 + \beta_2$ . If condition (8) holds, this implies that a given unemployment gap is associated with a larger output gap in the 1<sup>st</sup> regime, relative to the 2<sup>nd</sup>. Accordingly, for a given unemployment gap, we can think of regime “1” with the larger in absolute magnitude in  $\beta$  as the ‘steeper’ regime, while regime “2” with the flatter.

For instance, when public debt is below its threshold of around 40% of GDP,  $\delta_d = 0.404$  in panel B, the output gap is about three times as large as the unemployment gap ( $\beta_1 = -3.217$ ). Otherwise, when debt exceeds its threshold, they move around one-to-one ( $\beta = -0.974$ ). Consider also female labor force participation. This has been rising through most of the sample, boosting growth and countering the decline in male participation. Accordingly, any unemployment falls over this period would have had a more expansionary effect on cyclical output relative to the period (from the mid-1990s onwards) when that increase abated and reversed: i.e.,  $\beta_1 = -1.967$  vs.  $\beta = -1.417$ . Indeed, several studies have found Okun’s coefficient to be sensitive to female participation rates (e.g., Lee (2000)).

Results can be supplemented by examining the transition probabilities. Figures 2-3 plot the threshold time series,  $z$ , overlaid with threshold scalar  $\delta$ , and the transition probabilities  $g(z_t; \gamma, \delta)$ .<sup>22</sup> Note we see the graphical analogue of the speed of adjustment parameter values. The  $\gamma$  estimates are significantly different from zero but quite heterogeneous in value. The higher is  $\gamma$  the more rapid is the transition

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<sup>22</sup> The equivalent plots with the NBER recession dates overlaid are provided for reference in supplementary Figure C.2-Figure C.3.

between regimes.<sup>23</sup> High values, i.e.,  $\gamma \geq 10$ , as in  $u_n, pr_f$  and  $i_l$  often have volatile and rapid adjustment transition probabilities. The labor share and the debt ratios exhibit intermediate transitions, i.e.,  $\gamma \in [1, 10)$ . Finally, the lowest speeds adjustment speeds, i.e.,  $\gamma \in (0, 1)$ , are given by long-term unemployment,  $u_l$ , macroeconomic risk, the short rates, and manufacturing labor share.

Several interesting conclusions can be drawn from the transition probabilities. For instance, they imply both highly persistent (long lived) and less persistent (temporary) regime shifts of the OL. In the first category of shifts, we can include economic volatility and the debt series (whose transition occurs around mid to late 1980s), and the labor income share, manufacturing employment share and female participation rates (whose transition occurs in the mid to late 1990s).<sup>24</sup>

Second, the category of variables that imply short-term shifts of the Okun law consists of most of the policy/finance threshold variables (such as the interest rates, and perhaps also with the long term unemployed). These shifts and the transition probabilities are, like the series themselves, quite volatile with a cyclical (oscillatory) pattern and they do not impact immediately between the regimes. Regarding interest rates, the three rates point in the same direction. In the aftermath of the Great Recession, they identified a regime of substantially less steep Okun coefficients ( $\beta \approx -0.8, -0.9$ ). In other words, the unemployment and output gap mirrored one another almost one to one. The behavior of unemployment at the onset of the Great Recession is testament to that: unemployment rose more sharply after the financial crisis than would otherwise have been predicted.<sup>25</sup>

Firms may have understood early on the severity of the Great Recession and choose

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<sup>23</sup> This recalls the distinction made by Bernanke et al. (2005) who discuss ‘slow’ and ‘fast-moving’ variables.

<sup>24</sup> Regime wise, the natural rate is a little more opaque. The natural rate peaked in the late 1970s and fell monotonically thereafter (barring the period of the Great Recession). The start of the protracted fall from 1980s, though, traverses the (bootstrapped) upper confidence interval of the threshold.

<sup>25</sup> The rate of unemployment has also fallen more than the law would have predicted since the Great Recession.

to engage in less labor hoarding than otherwise (implying rising labor productivity), and perhaps substituted towards more capital-intensive production. Thus, this regime of more accommodative monetary policy was associated with substantially flatter Okun's trade offs. These results would speak to the possible state dependence of monetary and fiscal policy.<sup>26</sup>

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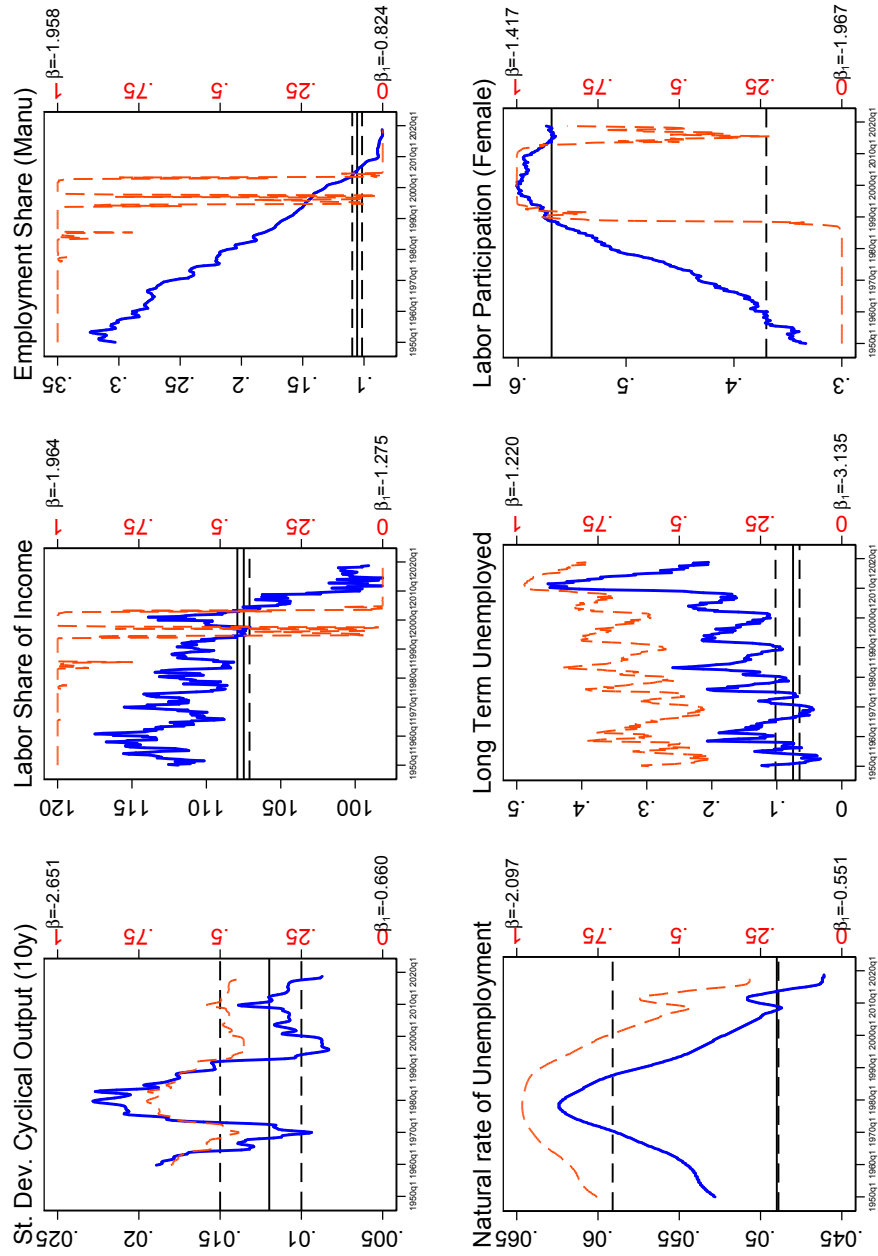
<sup>26</sup> More generally, Auerbach and Gorodnichenko (2012) found that for the US the size of the fiscal multiplier appeared to be state dependent; thus the state of the economy and policy effectiveness are innately linked. Moreover there is a growing literature on state dependency in monetary policy, particularly around the effective lower bound (see Woodford, 2012).

TABLE 3: Threshold Okun Law Results

	$labsh^\dagger$	$u_n$	$pr_f$	$u_l$	$d$	$\sigma_{y_{t(10)}}^c$	$i_s$	$i_{s,xw}$	$dp$	$i_l$	$lmanu$
<b>Ignoring Endogeneity</b>											
$\beta_1$	-1.246 (0.097)	-0.760 (0.206)	-1.967 (0.076)	-2.194 (0.117)	-2.562 (0.251)	-1.593 (0.107)	-1.413 (0.091)	-1.414 (0.091)	-2.024 (0.081)	-0.876 (0.185)	-0.926 (0.165)
$\beta_2$	-0.714 (0.122)	-1.355 (0.283)	0.519 (0.112)	1.339 (0.277)	1.659 (0.504)	-0.923 (0.251)	-0.651 (0.147)	-0.651 (0.147)	0.575 (0.126)	-1.243 (0.256)	-1.035 (0.197)
$\beta$	-1.960	-2.115	-1.448	-0.855	-0.903	-2.516	-2.064	-2.065	-1.449	-2.119	-1.961
$\delta$	107.519 {107.500,107.8210}	0.050 {0.0491,0.0591}	0.059 {0.037,0.591}	26.100 {16.100,25.200}	0.625 {0.402,0.932}	2.078 {1.482,2.062}	4.310 {4.060,4.460}	4.340 {4.001,4.560}	34.160 {34.100,32.200}	0.004 {0.0025,0.084}	10.562 {10.157,10.958}
$\gamma$	5.5 {4.10,7.30}	303.9 {301.20,307.26}	340.0 {300.00,341.00}	0.1 {0.09,0.30}	3.5 {0.10,4.00}	3.3 {1.50,5.00}	2.0 {1.01,3.452}	2.0 {1.30,3.50}	2.9 {1.50,3.40}	70.0 {68.50,71.50}	0.6 {0.30,2.60}
$\mathcal{D}_2$	75.00	83.69	40.94	10.51	34.78	11.79	52.14	51.75	52.04	75.32	80.14
$AIC$	-1006.02	-1004.51	-1002.56	-1002.96	-1000.19	-860.86	-934.79	-934.88	-976.13	-863.40	-1002.91
<b>Controlling for Endogeneity</b>											
$\beta_1$	-1.275 (0.124)	-0.551 (0.275)	-1.967 (0.974)	-3.135 (1.544)	-3.217 (1.596)	-0.660 (0.344)	-0.825 (0.389)	-0.872 (0.338)	-1.992 (0.894)	-0.878 (0.170)	-0.824 (0.145)
$\beta_2$	-0.689 (0.162)	-1.546 (0.475)	0.550 (0.278)	1.915 (1.052)	2.243 (1.180)	-1.991 (0.866)	-1.391 (0.543)	-1.336 (0.495)	0.551 (0.279)	-1.289 (0.650)	-1.134 (0.185)
$\beta$	-1.964	-2.097	-1.417	-1.220	-0.974	-2.651	-2.216	-2.208	-1.441	-2.167	-1.958
$\delta$	107.919 {107.10,107.921}	0.049 {0.0489,0.0591}	0.569 {0.369,0.591}	7.500 {6.500,10.200}	0.404 {0.402,0.822}	0.012 {0.010,0.015}	2.160 {0.160,2.960}	2.140 {1.010,2.910}	34.730 {34.700,34.800}	0.044 {0.035,0.075}	10.173 {9.657,10.557}
$\gamma$	5.1 {4.60,70.00}	309.1 {307.50,310.10}	340.0 {300.00,341.00}	0.1 {0.09,0.301}	3.0 {0.100,4.000}	0.9 {0.872,1.420}	0.4 {0.10,1.90}	0.4 {0.10,1.90}	2.2 {0.9,3.40}	61.7 {60.0,63.01}	0.6 {0.30,2.30}
$100\omega_1$	0.020 (0.019)	0.080 (0.043)	-0.004 (0.045)	0.300 (0.207)	0.100 (0.043)	0.100 (0.039)	0.070 (0.039)	0.050 (0.026)	-0.010 (0.005)	0.060 (0.051)	0.040 (0.031)
$100\omega_2$	-1.280 (0.676)	-0.009 (0.005)	1.260 (3.500)	0.200 (0.168)	-0.010 (0.010)	-1.000 (0.586)	-0.700 (0.330)	-0.700 (0.325)	-0.700 (0.400)	-0.900 (0.473)	0.020 (0.159)
$\mathcal{D}_2$	73.18	89.85	40.94	86.96	90.22	58.96	75.48	75.09	51.67	67.96	80.14
$AIC$	-1018.72	-1003.24	-1000.41	-1008.10	-1000.01	-862.43	-936.05	-936.41	-974.22	-865.16	-1004.67

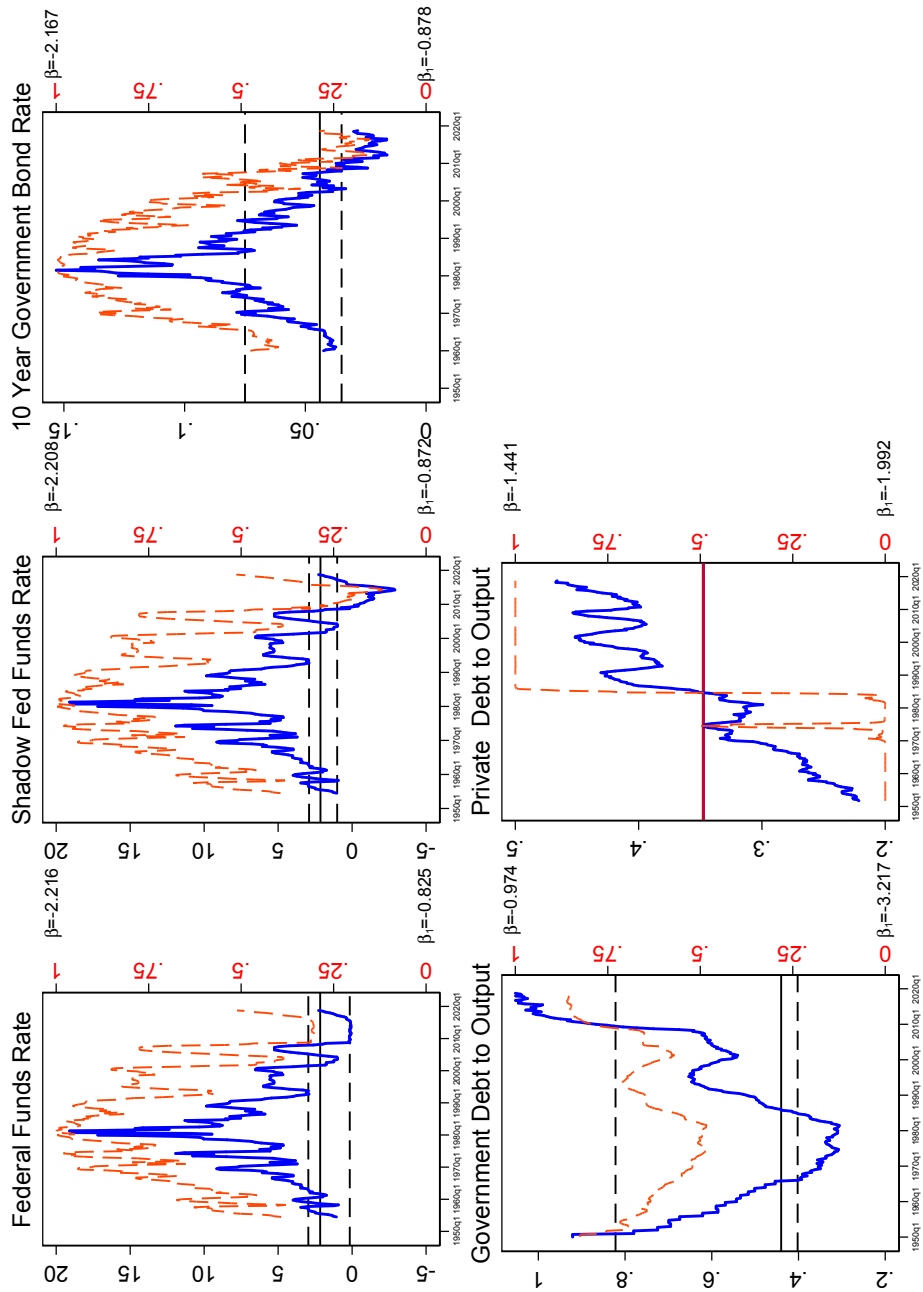
Notes: <sup>†</sup> Recall that the variant of the labor share that we use is the quarterly indexed one provided by FRED where 2012 = 100.

FIGURE 2: Threshold Variables, Threshold and Transition Probabilities (Structural Variables)



Notes: Blue solid lines represent the threshold variable,  $z$ , red dashed lines represent the  $\{0, 1\}$  transition probabilities and the solid black horizontal line represents the threshold,  $\delta$  (with black dashed confidence intervals). The  $\beta$  values on the rhs axis are the Okun parameters in the respective regimes.

FIGURE 3: Threshold Variables, Threshold and Transition Probabilities (Policy and Financial)



Notes: See notes to Figure 1.

Finally, in terms of comparing exogenous-endogenous threshold results, female labor force participation and employment in manufacturing are revealed as exogenous threshold types ( $\omega$  coefficients in these equations are insignificant). This is perhaps not so surprising. Increasing trend female labor participation, though doubtless influenced by economic incentives in many ways, is generally regarded more as a cultural and demographic phenomenon, Bullard (2014). Likewise, the declining employment and value added share of manufacturing is a long standing trend, reflecting changes in global trading relations and shifting comparative advantages etc.

## 7 A Composite Threshold Variable

Our analysis suggests that many of the regime shifts triggered by the threshold variables examined co-move. This is true for both long-lived and more temporary shifts. [Table 4](#) corroborates this showing the extent of the correlations among the  $\{0, 1\}$  transition probabilities across regimes for all  $z$  candidates (all cross-correlations turn out to be significant at 1%). This co-movement can be attributed to common, or related, sources of information and economic events underlying the threshold variables triggering the threshold effects in OL. As we saw, these appear mostly related to structural and/or financial policy changes.

Accordingly, we now combine these different information sources into a single common factor and, then, (re-)estimate threshold model (7). The threshold factor effects considered can summarize and smooth out all the alternative sources of the OL regime shifts, which are related, and can potentially better describe and account for the total effects of such shifts over time. Consider the  $K$ -dimension column vector of the significant threshold variables considered, defined as  $Z = [z_j]$ ,  $j = 1, 2, \dots, K$ . For notational convenience, we drop time-subscripts from  $z_{jt}$ . This then implies that the PCs, or factors, collected in column vector  $p$ , can be written as

$$C_i = p'_i Z, \quad i = 1, 2, \dots, K$$



TABLE 4: Correlation of the Transition Probabilities

	<i>labsh</i>	<i>u<sub>n</sub></i>	<i>pr<sub>f</sub></i>	<i>u<sub>l</sub></i>	<i>d</i>	$\sigma_y^2$	<i>i<sub>s</sub></i>	<i>i<sub>s,wx</sub></i>	<i>dp</i>	<i>i<sub>l</sub></i>	<i>lmanu</i>
<i>labsh</i>	1										
<i>u<sub>n</sub></i>	0.84	1									
<i>pr<sub>f</sub></i>	-0.57	-0.59	1								
<i>u<sub>l</sub></i>	-0.71	-0.66	-0.49	1							
<i>d</i>	-0.75	-0.81	0.58	0.74	1						
$\sigma_y^2$	0.54	0.62	-0.68	-0.23	-0.55	1					
<i>i<sub>s</sub></i>	0.66	0.78	-0.42	-0.72	-0.76	0.49	1				
<i>i<sub>s,wx</sub></i>	0.67	0.77	-0.43	-0.74	-0.78	0.47	0.99	1			
<i>dp</i>	-0.59	-0.63	0.84	0.54	0.71	-0.58	-0.42	-0.41	1		
<i>i<sub>l</sub></i>	0.67	0.82	-0.36	-0.56	-0.73	0.53	0.90	0.91	-0.32	1	
<i>lmanu</i>	0.88	0.94	-0.53	-0.75	-0.81	0.50	0.81	0.79	-0.60	0.80	1

Notes: The table presents estimates of the correlation coefficients of the transition probabilities from regime 1 to “2” for variables  $z_j$  which trigger significant threshold effects.

where  $p'_i$  is the  $i^{th}$ -row of an orthogonal matrix  $P = [p_{ij}]$  (normalized, i.e.,  $P'P = I$ ) such that  $P'\Sigma_z P = \Lambda = diag[\lambda_i]$ , with  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_K \geq 0$ , where  $\Sigma_z$  is the covariance matrix of vector  $Z$  and  $\Lambda$  is the matrix of the eigenvalues of  $\Sigma_z$ , denoted  $\lambda_i$ , by the spectral decomposition theorem. Note that the first factor  $C_1$  (which, to recall, accounts for the single largest part of the data variation) corresponds to the largest eigenvalue  $\lambda_1$ . The covariance and correlation loading coefficients between  $C_i$  and a threshold variable  $z_j$ ,  $j = 1, 2, \dots, K$ , collected in  $Z$ , are respectively given by

$$cov(C_i, z_j) = p_{ij} \sqrt{\lambda_i} \quad \text{and} \quad \rho_{ij} = \frac{p_{ij} \sqrt{\lambda_i}}{\sigma_{z_j}}$$

Note, the correlation loading coefficients are net of the effects of differences in the variances of series  $z_j$ . The proportion of the variation of the data accounted for by the first  $v$ -PCs is given as  $\rho_v^2 = \frac{\sum_{i=1}^v \lambda_i}{\sum_{i=1}^K \lambda_i}$ . To formally examine if the first  $v$  PCs are non trivial (i.e., reflect a systematic pattern of variation of the data providing a meaningful interpretation of the data) we apply some testing procedures suggested in the literature (see Peres-Neto et al. (2005)).

## 7.1 Results

**Table 5** reports estimates of model (7) based on a PC factor as the threshold variable.<sup>27</sup> Bootstrap standard errors are in parentheses and confidence intervals in braces. **Panel A** is based on standardized deviations of the variables of vector  $Z$  from their mean (i.e.,  $\tilde{z}_j = \frac{z_j - \mu_{z_j}}{\sigma_{z_j}}$ ), while **Panel B** uses transformation  $\ddot{z}_j = \frac{z_j - z_j^{\min}}{z_j^{\max} - z_j^{\min}}$ . Both transformations re-scale the original variables  $z_j$  to avoid the influence of differences in the measurement units and quantities of  $z_j$ 's.

The PC analysis based on standardized variables  $\tilde{z}_j$  has the following useful property. It weights equally the contribution of the variance of each series  $z_j$  into the variation  $C_i$ , otherwise variables with a larger variance will have larger  $cov(C_i, z_j)$  and  $\rho_{ij}$  on the first PC  $C_1$  (see, e.g., Timm (2002)). Yet, most of the tests determining the number of the non-trivial PC rely on standardized data.

The second rescaling method, based on  $\ddot{z}_j$ , has also two main interesting and novel features: (1) it also makes results independent of the measurement units; (2) it provides an indexed-factor between 0 and 1, which may aid interpretability. Note that, in contrast to standardized variables  $\tilde{z}_j$ ,  $\ddot{z}_j$  will not have the same variance for all  $j$ . Although the variance of  $\ddot{z}_j$  will be less than unity, it can differ across  $j$ . This is a useful property of transformation  $\ddot{z}_j$ , if it is meaningful economically to weight more the variables of the PC  $C_1$  with higher variability. This transformation can achieve this, without depending on differences in the scale across the original variables  $z_j$ .

Regarding the  $AIC$ , the versions of the threshold model with these two scaled threshold variables constitute superior specifications of the data compared to those of **Table 3**. Results suggest the following. First, the values of the  $LR^{\text{sup}}$  validate that there are significant threshold effects in OL. Second, the coefficients  $\beta_1$  and  $\beta$  demonstrate that there are substantial differences in the OL slope between its two regimes and that

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<sup>27</sup> To check robustness with respect to the frequency of the data, we re-ran our exercises on annual data. Results were qualitatively similar. The longer regime has an OL coefficient of around  $-2$  and  $-1$  otherwise. The estimates are shown in supplementary material **Table D.1**. The corresponding PCs and transition functions are plotted in supplementary **Figure D.1**.

the speed of transition between regimes is quite high:  $\gamma \approx 10$ .

For both scaling methods of  $z$ , we present estimates of the eigenvalues  $\lambda_i$  and the proportion of variation  $\rho_v^2$ , for  $v = 1, 2, \dots, K$ , the correlation loading coefficients  $\rho_{ij}$  for  $C_1$  and the model estimates considering  $C_1$  as the threshold variable. This first factor is found to be the only non-trivial factor spanning the data, based on the Broken-Stick and Auer-Gervini methods determining the number of factors to retain.<sup>28</sup> Further support that  $C_1$  can sufficiently summarize the related information of the data can be obtained by the much higher value estimates of  $\lambda_i$  for  $C_1$ , compared to the remaining PC factors reported in the table. The  $\rho_{v=1}^2$  estimates indicate that  $C_1$  explains roughly 70% of the data variability, while the estimates of  $\rho_{ij}$  imply that all variables  $z_j$  contribute to the sample variation of  $C_1$  and are substantially correlated with it. The estimates of  $\rho_{ij}$  for the indexed-factor  $C_1$  (see Panel B), which weights more the series  $z_j$  with the higher variation indicate that, among all variables  $z_j$ , those having the higher correlation and contribution to the variation of  $C_1$  are the following: *lmanu*, *u<sub>n</sub>*, *d*, *dp* and *pr<sub>f</sub>* (cells marked in gray).

Figure 4 demonstrates that both scaling methods provide smooth and broadly consistent patterns of movements of factor  $C_1$  and the accompanying transition probabilities. In fact, they indicate that there essentially exists only one regime transition from a steeper to a flatter OL. The steeper slope lasts until the end of 80s (early 90s), with high probability (almost 90%). After this, there is a tendency to move to a flatter Okun relationship, which is noticeably more likely after the financial crisis. The threshold is less precisely estimated under the first scaling method, although the fact that the transition to a flatter OL traverses the upper confidence limit for  $\delta$  is also interesting and informative.

Third, the positive sign of the estimates of loading coefficients  $\rho_{1j}$ , indicate that important sources of the steeper OL relationship (more likely in 70s and 80s) are the

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<sup>28</sup> These methods are found to work satisfactorily in a number of simulation studies (see Peres-Neto et al (2005), for a survey). The Broken-Stick test statistic compares the variance of factors  $C_i$  with the values expected from the broken-stick distribution, while the Auer-Gervini method is Bayesian and relies on the model selection criterion (see Auer and Gervini (2008)).

higher levels of labor income share, manufacturing employment share, the natural rate of unemployment and interest rates. On the other hand, important sources of occurrence of a more flat Okun relationship and a transition to such a regime are the following: the further increase of the public and private debts since 90's and middle 80's, respectively, the decline of policy interest rates, especially after 2000, and the slight positive trend of long run unemployment since the 80s (plus female labor participation).

These can be justified by the negative sign of the estimates of  $\rho_{1j}$ . These results hold for both re-scaling methods. They are also consistent with our earlier estimates of Table 3. This means that the threshold factor  $C_1$  can efficiently summarize the sources of the OL relationship threshold effects examined, over time. This factor may be thus prove a useful policy tool in indicating possible shifts in the relationship between unemployment and economic activity and taking appropriate actions to offset adverse effects of such shifts.

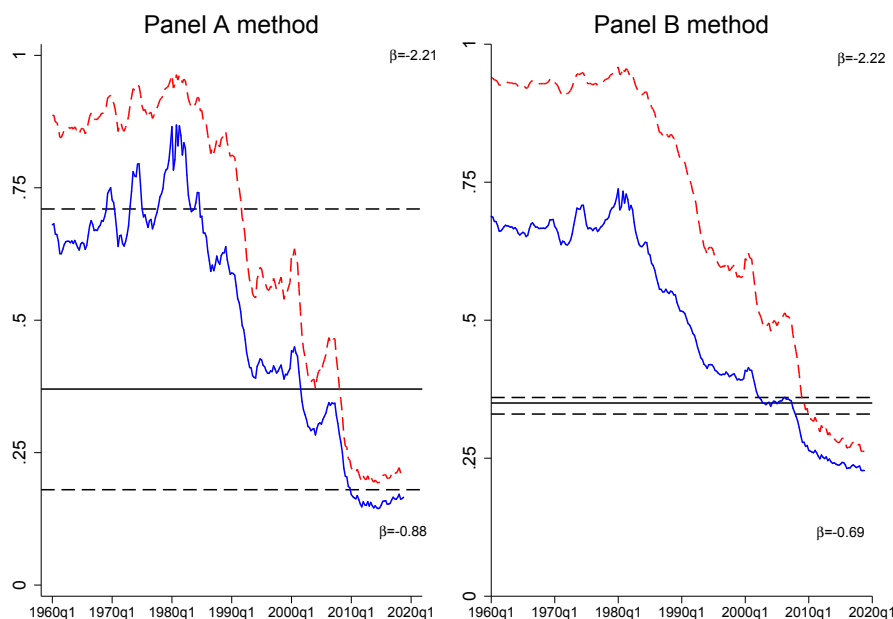
Finally, regarding the issue of endogeneity of factor  $C_1$ , results demonstrate that ignoring this leads to different estimates of the parameters of the threshold model, especially the  $\beta$ s and  $\gamma$ . The estimates of  $\gamma$  and their standards errors imply that this issue is important, especially in regime "2", where  $\lambda_2$  is clearly different than zero at 5% (or less). Further support for estimating dealing with the endogeneity of factor  $C_1$  can be obtained by the values of  $AIC$  reported in the table. These show that the versions of the model controlling for the endogeneity of  $C_1$  fit better the data, compared to those that do not.

TABLE 5: PC analysis and model estimates based on the threshold common-factor variable

<b>Panel A: Estimates based on standardized variables <math>\tilde{z}_j</math></b>											
$i, v$	1	2	3	4	5	6	7	8	9	10	11
$\lambda_i$	7.1800	2.0200	0.9300	0.3200	0.2000	0.1300	0.0800	0.0600	0.0300	0.0060	0.0040
$\rho_i^2$	0.6500	0.8400	0.9300	0.9500	0.9700	0.9800	0.9900	0.9950	0.9990	0.9996	1.0000
$j$	<i>labsh</i>	$u_n$	<i>prf</i>	$u_t$	$d$	$\sigma_y^2$	$i_s$	$i_{s, var}$	$dp$	$i_t$	<i>lmanu</i>
$\rho_{ij}$	0.31	0.35	-0.23	-0.28	-0.35	0.26	0.30	0.31	-0.28	0.28	0.32
	$\beta_1$	$\beta_2$	$\beta$	$100\omega_1$	$100\omega_2$	$\gamma$	$\delta$	$D_2$	$AIC$	$LM^{sup}$	
Ignoring Endogeneity											
	-1.02	-1.09	-2.11			8.80	0.36	70.62	-867.32	[0.000]	
	(0.15)	(0.21)				(5.10,9.10)	(0.35,0.37)				
Controlling for endogeneity											
	-0.88	-1.33	-2.21	0.03	-0.7	6.90	0.37	70.60	-868.70	[0.000]	
	(0.17)	(0.25)		(0.03)	(0.30)	(4.00,9.30)	(0.18,0.71)				
<b>Panel B: Estimates based on scaled variables <math>\tilde{z}_j</math></b>											
$i, v$	1	2	3	4	5	6	7	8	9	10	11
$\lambda_i$	0.5200	0.1300	0.0600	0.0200	0.0100	0.0080	0.0060	0.0030	0.0020	0.0005	0.0001
$\rho_i^2$	0.6800	0.8400	0.9200	0.9500	0.9700	0.9800	0.9900	0.9960	0.9990	0.9998	1.0000
$j$	<i>labsh</i>	$u_n$	<i>prf</i>	$u_t$	$d$	$\sigma_y^2$	$i_s$	$i_{s, var}$	$dp$	$i_t$	<i>lmanu</i>
$\rho_{ij}$	0.27	0.38	-0.34	-0.22	-0.37	0.28	0.18	0.17	-0.34	0.19	0.42
	$\beta_1$	$\beta_2$	$\beta$	$100\omega_1$	$100\omega_2$	$\gamma$	$\delta$	$D_2$	$AIC$	$LM^{sup}$	
Ignoring endogeneity											
	-0.17	-1.96	-2.13			8.10	0.25	90.03	-866.68	[0.000]	
	(0.30)	(0.38)				(7.20,8.40)	(0.24,0.27)				
Controlling for endogeneity											
	-0.69	-1.53	-2.22	0.04	-0.50	8.20	0.35	72.41	-868.10	[0.000]	
	(0.18)	(0.38)		(0.03)	(0.20)	(6.25,8.90)	(0.33,0.36)				

Notes: The table presents results of the PCA of the variables  $z_j$  triggering significant threshold effects and estimates of model (7) using the first PC factor  $C_1$ , as the threshold variable. These estimates cover the case that the endogeneity of  $C_1$  is ignored and is controlled based on our suggested copula method. Panel A presents results for the case that the PCA analysis is based on standardized transformation of  $z_j$ , i.e.,  $\tilde{z}_j = \frac{z_j - \mu_{z_j}}{\sigma_{z_j}}$ , while Panel B for transformation  $\tilde{z}_j$ , thus scaling  $z_j$  in the interval [0,1]. Bootstrap standard errors and confidence intervals are, respectively, given in parentheses and braces, whilst the  $LM^{sup}$  probability values are given in brackets.

FIGURE 4: Transition Probability (Principal Components)



Notes: This figure graphically presents estimates of the factor  $C_1$  against the transition probability of the model using  $C_1$  as the threshold variable. This is done for the two methods scaling the data. The first is based on standardized variables  $\tilde{z}_j = \frac{z_j - \mu_{z_j}}{\sigma_{z_j}}$ , while the second is based on  $\tilde{z}_j = \frac{z_j - z_j^{\min}}{z_j^{\max} - z_j^{\min}}$ , scaling  $z_j$  in the interval  $[0, 1]$ . As before, the series in blue is the PC, and in red dashed we have the transition probabilities. See also notes to [Figure 1](#).

## 8 Conclusions

Okun's law is a building block of many macro-econometric models and often considered an empirical regularity. We examined the possibility of asymmetries in that relationship. Specifically, we considered a nonlinear, smooth transition regression model, exploring a variety of candidate threshold variables. Our paper makes two main contributions. The first is methodological, the second is empirical.

In terms of **methodology**, our work has two distinct components:

- (a) We allow for endogeneity in the threshold variable using copulas to capture the dependence between the disturbance term and the threshold variable in the context of a LSTR model. The copula approach admits normal and non-normal dependencies, thus adding an additional flexibility in modelling the endogeneity.
- (b) We suggest a *new* testing procedure to test for smooth transition threshold effects against linearity under endogeneity of the threshold variable.

Both the copula based approach and the linearity testing procedure suggested are evaluated throughout a MC study.

Regarding **empirics**, which is our second main contribution, we establish that threshold effects can be detected in Okun's relationship. We find a combination of structural and policy-related variables accounts for changes in the Okun's law trade off in recent decades. This conclusion is bolstered by combining these threshold candidates into a single factor. We found regime-like behavior with Okun's coefficient rising (or flattening) over time: from around  $\leq -2$  over the 1960s-1980s, then a slow transition to a value around  $\geq -1$ . Thus the unemployment gap is increasingly associated with a smaller output gap. Moreover, whilst the Great Recession accelerated that rise, interestingly, the bulk of the change occurred beforehand. This, in turn, corroborates our finding that both structural and non structural factors were at play. Of our candidates, special emphasis lies with manufacturing employment, female participation, the natural rate of unemployment and the private and public debt series.

The work done here, moreover, could be extended. An interesting controversy in the literature is whether OL holds outside the US and in particular for countries characterized by less flexible labor and product markets. More generally, though, our endogenous threshold LSTR model could prove useful for additional applications such as in studies of growth, trade and finance where threshold models have often been used to analyze asymmetries (albeit in a more restrictive form than that considered here) and may thus yield new or more robust insights.

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## APPENDICES

### A Monte Carlo

In this section, we present the data generating processes (DGP) and results of the Monte Carlo (MC) exercise touched upon in the main text. Our first exercise examines the performance of the estimation method suggested to estimate the location parameter  $\delta$  accurately and the ability of our method to successfully control for the endogeneity of threshold variable  $z_{it}$  in the LSTR model. The second exercise (in [Appendix B](#)) evaluates the power performance of test statistic  $LR^{\text{sup}}$ .

The DGP that we consider to estimate  $\delta$  is as follows:<sup>1</sup>

$$y_t = (\beta_{11} + \beta_{12}x_{2t} + \beta_{13}x_{3t})(1 - g(z_t; \gamma, \delta)) + (\beta_{21} + \beta_{22}x_{2t} + \beta_{23}x_{3t})g(z_t; \gamma, \delta) + \varepsilon_t, \quad (\text{A.1})$$

where  $x_{2t} \sim IIDN(0, 1)$  and  $x_{3t} \sim IIDN(0, 1)$  are exogenous, i.e.,  $\mathbb{E}(\varepsilon_t | x_{2t}) = \mathbb{E}(\varepsilon_t | x_{3t}) = 0$ , and the values of the slope coefficients across the two regimes of the model are given as follows:

$$\text{Regime 1: } \beta_{11} = 0.5, \quad \beta_{12} = 2, \quad \beta_{13} = 1,$$

$$\text{Regime 2: } \beta_{21} = -0.7, \quad \beta_{22} = \{1.5, 3.0\}, \quad \beta_{23} = 0.$$

These values imply quite small differences in the slope coefficients of model (A.1) across its two regimes, which are difficult to detect. They are chosen to highlight the ability of our method to lead to accurate estimates of  $\delta$  even for very small in size shifts of the slope coefficients of the model.

For the regression error term  $\varepsilon_t$  and threshold variable  $z_t$  of the above mentioned DGP, we consider the normal and the Student-t distribution, with four degrees of freedom. Using data from the Student's  $t$  distribution will show if our method, based on the transformations of the quantile function of the standard normal distribution, i.e.,  $\Phi^{-1}$ , can be proved robust to a misspecification of the true distributions of  $\varepsilon_{h,t}$  and  $z_t$ , like the Student's  $t$  often assumed in econometrics. For reasons of space, we present results for the cases that the distribution of  $\varepsilon_t$ , and correlation structure between  $\varepsilon_{h,t}$  and  $z_t$  do not change across the two regimes of the model. For  $z_t$ ,

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<sup>1</sup> Both the DGP and the values of its parameters considered in our simulation analysis are close to those considered in the simulation studies of, for instance, Lundbergh, Teräsvirta and van Dijk (2003).

we assume  $z_t \sim IID(z_\mu, 1)$ , with  $z_\mu = 3.0$ , while, for the correlation coefficient between  $\varepsilon_{h,t}$  and  $z_t$ , we consider the following set of values:  $\rho_{\varepsilon z} = \{0.0, 0.55, 0.75\}$ , for  $h = \{1, 2\}$ . The structure of the threshold variable is given as follows:

$$z_t = \delta + c_{\varepsilon z} v_t + \zeta_t \quad (\text{A.2})$$

where  $\varepsilon_t \sim t_{DF=4}$  and  $\zeta_t \sim t_{DF=4}$ , for all  $t$ . The value of coefficient  $c_{\varepsilon z}$  is chosen to control for the degree of correlation between  $\varepsilon_{it}$  and  $z_{it}$

The threshold value  $\delta$  is set to the 25% percentile (1<sup>st</sup> quantile) of the distribution of  $z_t$ . We also examined threshold values at the 75% percentile (3<sup>rd</sup> quantile) of this distribution, but the MC results do not change qualitatively. The values of the speed of adjustment considered are set to  $\gamma = \{1.5, 3.5\}$ . These values reflect the cases that the transition between the two regimes is low and high, respectively. Following other studies, we treat the above values of  $\gamma$  as known in our analysis, reflecting that our interest is focused on the estimation bias of  $\delta$ . Since we are focused on the performance of our method to control for the endogeneity of  $z_t$  on the estimates of  $\delta$ , we do not report results for the remaining parameters of the model, i.e.,  $\omega_1, \omega_2$  for reasons of space.

We consider sample sizes of  $T = \{50, 250\}$  observations and carry out 1,000 iterations. For all iterations, we calculate the bias and the root mean square error of the estimator of  $\delta$ . In [Table A.1](#), we present average values of the above metrics, over all iterations, denoted as BIAS and RMSE, respectively. The table presents different sets of results. [Panel A](#) presents results ignoring the problem of the threshold variable endogeneity in the estimation. [Panel B](#) controls for this problem, based on our method by including in the rhs of [\(A.1\)](#) the bias correction terms  $z_{h,t}^*$ ,  $h = \{1, 2\}$ .

[Panel C](#) presents results for the case that Kourtellos's et al. (2016) approach for threshold models, appropriately modified for the LSTR model, is employed to control for the endogeneity problem of  $z_t$ . This approach adjusts equation [\(A.1\)](#) by estimating the expectation terms  $\mathbb{E}(\varepsilon_t | z_t \leq \delta)$  and  $\mathbb{E}(\varepsilon_t | z_t > \delta)$ , capturing the bias of  $z_t$  across the two regimes of the model, based on the inverse Mills ratio terms assuming that both  $\varepsilon_t$  and  $z_t$  are normally distributed. To calculate these ratios, we assume that  $z_t \sim IIDN(z_\mu, 1)$  has the following single factor presentation:

$$z_t = z_\mu + 0.95\zeta_t + e_{zt}, \quad (\text{A.3})$$

where  $\zeta_t \sim IIDN(0, 0.5)$  and  $e_{zt} \sim IIDN(0, 0.5)$  and  $z_\mu = 3.0$ . Finally, [Panel D](#)

presents results for the case that our method is applied to the case that  $\varepsilon_{h,t}$  and  $z_t$  are jointly Student- $t$  distributed.

Results lead to several interesting conclusions. First, they indicate that ignoring the endogeneity of threshold variable  $z_t$  causes serious biases in the estimates of threshold parameter  $\delta$ . Specifically, it tends to overestimate the true value of  $\delta$ . Moreover, as expected, the bias of  $\delta$  is substantial in magnitude when the correlation between  $z_t$  and  $\varepsilon_t$  is high, i.e., 0.75. The magnitude of the bias also depends on the different values of  $\gamma$  considered. The bias is bigger, the smaller value of  $\gamma$  considered, i.e.  $\gamma = 1.5$ , than  $\gamma = 3.5$ , and it remains even if  $T$  increases.

A second conclusion is that our method can successfully control for the endogeneity problem of the threshold variable. The method can substantially reduce the estimation bias of  $\delta$ . This is true even if the sample size is small (i.e.,  $T = 50$ ). As expected, the bias of  $\delta$  reduces as the sample increases (i.e.,  $T = 250$ ). The bias also reduces, when  $\gamma$  increases (i.e., the LSTR model approaches the TAR, where the shifts of the model are faster across the two regimes). Similar conclusions to the above also hold for the RMSE. These results are also robust to the different distribution of  $\varepsilon_{h,t}$  and  $z_t$  considered, namely the Student's  $t$ . This is clearer for the case of the larger size of  $T = 250$ . Note that, for the case that both  $\varepsilon_{h,t}$  and  $z_t$  are normally distributed, the performance of our method successfully compares to that of KST. Actually, our method seems to have better small sample (i.e.,  $T = 50$ ) performance relative to KST.

Finally, the results indicate that the estimation of the augmented regression model, which controls for the endogeneity of the threshold variable, leads to unbiased estimates of  $\delta$  even if there is no threshold variable endogeneity (i.e.,  $\rho_{\varepsilon z} = 0$ ). This result is more clear cut for the cases that  $T$  is large.

## References

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TABLE A.1: Monte Carlo results of the BIAS and RMSE of the estimator of  $\delta$

$\gamma$	$T$	$\rho_{\varepsilon z} = 0.0$		$\rho_{\varepsilon z} = 0.55$		$\rho_{\varepsilon z} = 0.75$		$\rho_{\varepsilon z} = 0.0$		$\rho_{\varepsilon z} = 0.55$		$\rho_{\varepsilon z} = 0.75$	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
(i) $b_{22} = 1.5$													
<b>Panel A: Ignoring the endogeneity of <math>z_t</math> and <math>\varepsilon_t</math> are normally distributed</b>													
1.5	50	-0.613	1.732	-0.975	2.986	-1.002	3.238	-0.811	2.420	-0.990	3.257	-0.991	3.464
1.5	250	-0.213	0.417	-0.827	1.697	-1.043	2.113	-0.153	0.324	-0.529	0.982	-0.716	1.250
3.5	50	-0.398	0.961	-0.816	2.120	-0.929	2.533	-0.331	1.197	-0.647	1.651	-0.694	1.879
3.5	250	-0.020	0.057	-0.439	0.899	-0.782	1.528	-0.012	0.049	-0.174	0.343	-0.358	-0.592
<b>Panel B: Controlling for the endogeneity of <math>z_t</math> and <math>\varepsilon_t</math> are normally distributed</b>													
1.5	50	-0.775	1.611	-0.729	1.829	-0.567	1.799	-0.732	1.522	-0.650	1.588	-0.485	1.561
1.5	250	-0.456	0.662	-0.343	0.614	-0.293	0.647	-0.381	0.546	-0.245	0.425	-0.165	0.421
3.5	50	-0.619	1.331	-0.503	1.287	-0.409	1.289	-0.522	1.098	-0.424	1.007	-0.266	0.884
3.5	250	-0.149	0.204	-0.135	0.204	-0.114	0.167	-0.073	0.111	-0.065	0.091	-0.072	0.092
<b>Panel C: Controlling for the endogeneity of <math>z_t</math> based on Mill's ratios (<math>z_t</math> and <math>\varepsilon_t</math> are normally distributed)</b>													
1.5	50	-0.891	2.363	-0.935	2.433	-0.928	2.423	-0.818	2.130	-0.863	3.202	-0.867	2.271
1.5	250	-0.398	0.811	-0.425	0.863	-0.428	0.873	-0.314	0.635	-0.315	0.661	-0.312	0.653
3.5	50	-0.691	1.559	-0.678	1.514	-0.667	1.500	-0.530	1.218	-0.572	1.295	-0.585	1.317
3.5	250	-0.091	0.189	-0.126	0.231	-0.129	0.234	-0.034	0.088	-0.057	0.125	-0.062	0.031
<b>Panel D: Controlling for the endogeneity of <math>z_t</math> and <math>\varepsilon_t</math> are Student-t distributed</b>													
1.5	50	-0.920	2.516	-0.993	2.615	-0.901	2.378	-0.878	2.425	-0.902	2.434	-0.846	2.214
1.5	250	-0.240	1.104	-0.516	1.381	-0.683	1.638	-0.133	0.847	-0.334	1.010	-0.518	1.295
3.5	50	-0.678	2.012	-0.696	1.954	-0.630	1.790	-0.599	1.838	-0.619	1.737	-0.499	1.393
3.5	250	-0.004	0.401	-0.231	0.508	-0.337	0.656	-0.035	0.218	-0.108	0.245	-0.150	0.271

Notes: The table presents average values of the *Bias* and *RMSE* of the estimates of  $\gamma$  and  $\delta$  estimates, over 1000 iterations, when both  $z_t$  and  $\varepsilon_t$  are normally and then Student-t distributed, with four degrees of freedom. This is done under different values of slope parameter:  $b_{22} = \{1.5, 3\}$  and the speed-of-transition coefficient:  $\gamma = \{1.5, 3.5\}$ . For the correlation coefficient between  $\varepsilon_t$  and  $z_t$ , we consider the following set of values:  $\rho_{\varepsilon z} = \{0.00, 0.55, 0.75\}$ , for both regimes of the model.



## B Power of the Test Statistic $LR^{\text{sup}}$

We present values of the power performance of statistic  $LR^{\text{sup}}$  for the LSTR model (A.1), employed in our previous MC exercise, where the threshold parameter value  $\delta$  is set at its 3<sup>rd</sup> quantile. For reasons of space, we do not report results for the case that  $\delta$  is at the 1<sup>st</sup> quantile and we consider only the case of  $\beta_{22} = 1.5$ . The remaining slope parameters are set as in the previous MC exercise. We also consider the same simulation scenarios with that exercise for  $\rho_{\varepsilon z} = \{0.0, 0.55, 0.75\}$  and  $T = \{50, 250\}$ .

To calculate the power of the test statistic, we need first to obtain its distribution under the null hypothesis  $H_0: \beta_1 = \beta_2 = \beta$  and obtain its critical value, corresponding to the 95<sup>th</sup>-quantile of its simulated distribution. To this end, for each iteration we generate the error term  $\varepsilon_t \sim IID(0, 1)$  and threshold variable  $z_t \sim IID(3, 1)$  from a joint distribution of them, while  $x_{2t}$  and  $x_{3t}$  are generated as  $x_{2t} \sim IID \mathcal{N}(0, 1)$  and  $x_{3t} \sim IID \mathcal{N}(0, 1)$ , respectively. Given these generated series, we then generate series  $y_t$  under null hypothesis based on model  $y_t = x_t' \beta + \varepsilon_t$ , where  $\beta_1 + \beta_2 = \frac{1}{2}(\beta_1 + \beta_2)$ ;  $\beta_1$  and  $\beta_2$  are defined as in the previous exercise.

Based on the generated series  $y_t$ , next we estimate model (A.1) under both the null and alternative hypotheses and, then, we calculate the test statistic  $LR^{\text{sup}}$ , over all possible values of  $\gamma$  and  $\delta$ , based on 1,000 iterations. In so doing, note that we trim out the top and bottom 10 percentiles of the distribution of  $\delta$ , while for  $\gamma$  we rely on the set of values  $Q_\gamma = \left[ \frac{1}{10\sigma_z}, \frac{100}{\sigma_z} \right]$ .<sup>2</sup> Also, under the alternative hypothesis, the model is adjusted by the bias correction terms  $z_{h,t}^*$ ,  $h = \{1, 2\}$  to control for the endogeneity of the threshold variable.

Given the critical values of  $LR^{\text{sup}}$ , at the 5% level, the power of the test, which represents rejection frequencies of the above null hypothesis, is calculated by generating data under the alternative hypothesis the alternative hypothesis  $H_a: \beta_1 \neq \beta_2$ , namely model (A.1). The remaining steps of the MC exercise are as above. For each iteration, the error term  $\varepsilon_t$ , the variables  $x_{2t}$  and  $x_{3t}$ , and the threshold variable  $z_t$  are generated as before, while in estimating  $\delta$  we trim out the top and bottom 10 percentiles of its distribution, while for  $\gamma$  we rely on the set of values  $Q_\gamma = \left[ \frac{1}{10\sigma_z}, \frac{100}{\sigma_z} \right]$ . To see if the power of the test depends on ignoring the issue of the endogeneity of the threshold variable  $z_t$ , we also present results for the case that model (A.1) is not adjusted by the terms  $z_{h,t}^*$ ,  $h = \{1, 2\}$  to calculate statistic  $LR^{\text{sup}}$ .

The results of the power of the test are reported in Table B.1. These indicate that

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<sup>2</sup> We found that extension of  $Q_\gamma$  to include higher values of  $\gamma$  does not affect our simulation results.



the power of the test statistic  $LR^{\text{sup}}$  is high and close to unity for all the cases of  $\gamma$  and  $\rho_{\varepsilon z}$  considered. They also indicate that ignoring the endogeneity of the threshold variable leads to a version of the test statistic which has less power. However, this is more clear for the case that  $\gamma = 3.5$ . Another interesting result is that the power performance does not depend on the adjustment of model (1) for possible endogeneity of threshold variable (see model (6)) when  $\rho_{\varepsilon z} = 0$ . This result was expected. Unless there is a serious degrees of freedom problem, it means that augmentation of this auxiliary regression by the transformed variables  $z_{ht}^*$  does not affect the power of statistic  $LR^{\text{sup}}$  and, hence, it can be safely implemented, in practice, for all possible values of the correlation coefficient  $\rho_{\varepsilon z}$ . The above results are robust to the different values of  $\gamma$  and  $T$  considered.

TABLE B.1: Power of test statistic  $LR^{\text{sup}}$  with  $\beta_{22} = 1.5$

	No		Controlling For	
	Endogeneity		Endogeneity	
$c_{\varepsilon z}$	$\gamma = 1.5$			
0.00	0.910	0.926	0.896	0.936
0.55	0.837	0.886	0.870	0.934
0.75	0.781	0.826	0.868	0.925
$c_{\varepsilon z}$	$\gamma = 3.5$			
0.00	0.852	0.893	0.827	0.896
0.55	0.766	0.833	0.812	0.896
0.75	0.687	0.762	0.804	0.895
$T$	50	250	50	250

Notes: The table presents the power of test statistic

$$LR^{\text{sup}} \equiv \sup_{(\gamma, \delta) \in Q_\gamma \times Q_\delta} LR(\gamma, \delta)$$

under the alternative hypothesis  $H_a: \beta^{(1)} \neq \beta^{(2)}$ , at the 5% nominal size, for model (A.1). The critical values of the test statistic are simulated under the null  $H_0: \beta^{(2)} = \beta^{(2)}$ , for alternative  $N$  and  $T$ . We use the parameter values from Appendix A but concentrate on the  $\beta_{22} = 1.5$  case.

# Supplementary Material For Internet Appendices

## C Additional Material

### C.1 Additional Figures

FIGURE C.1: Comparisons of FRED Labor Income Share Series

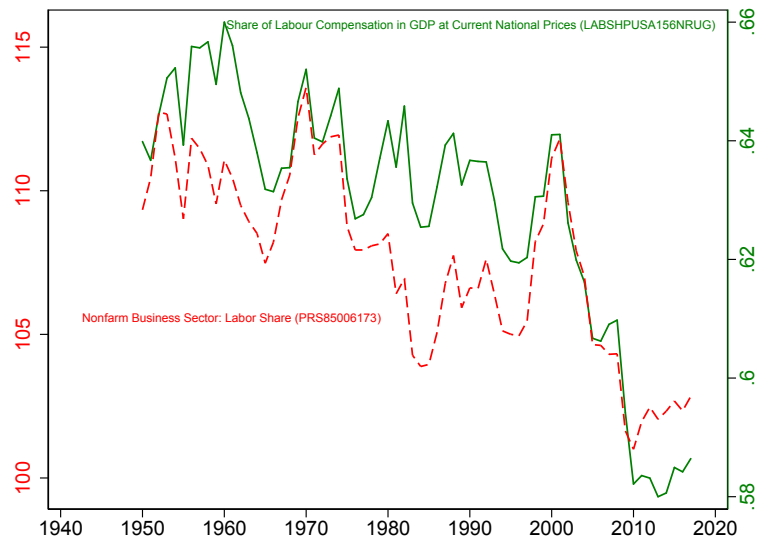
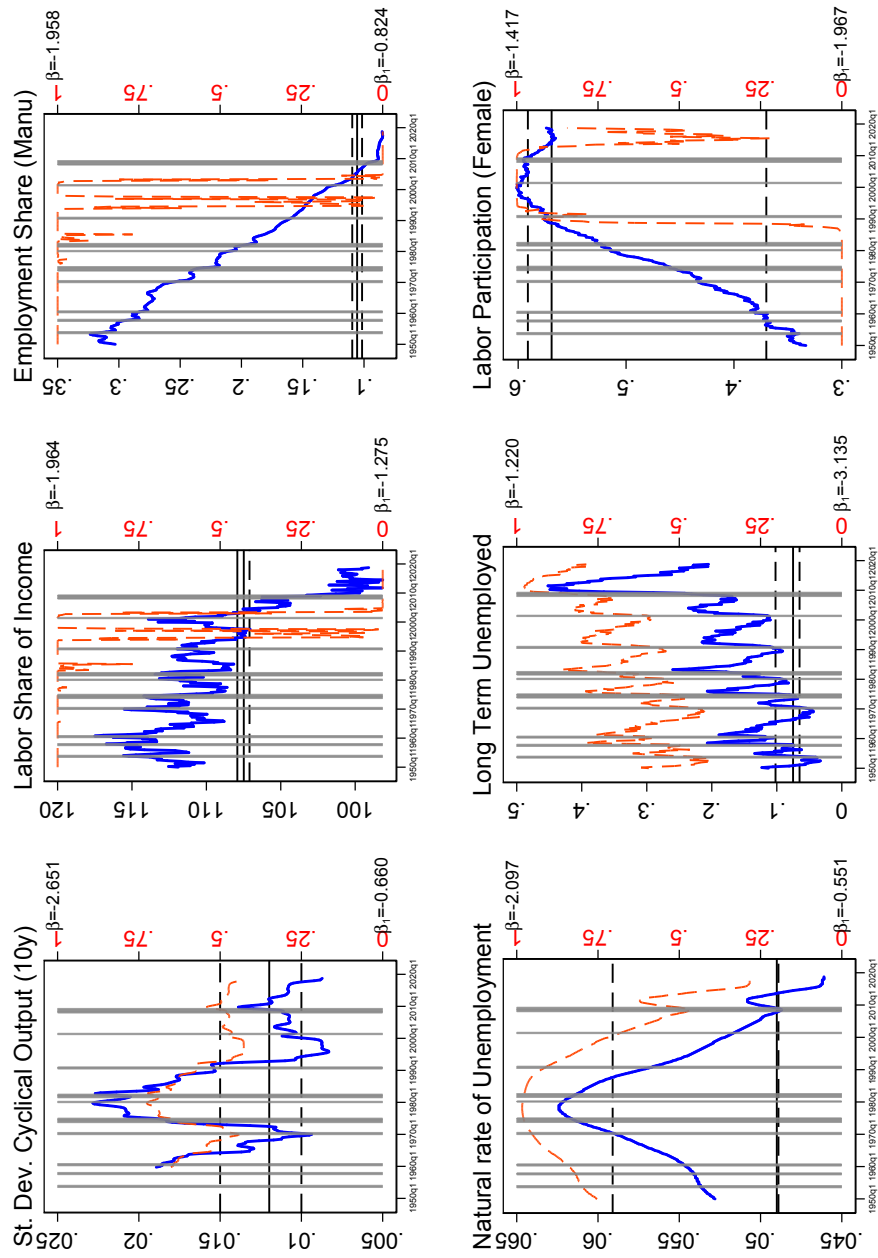
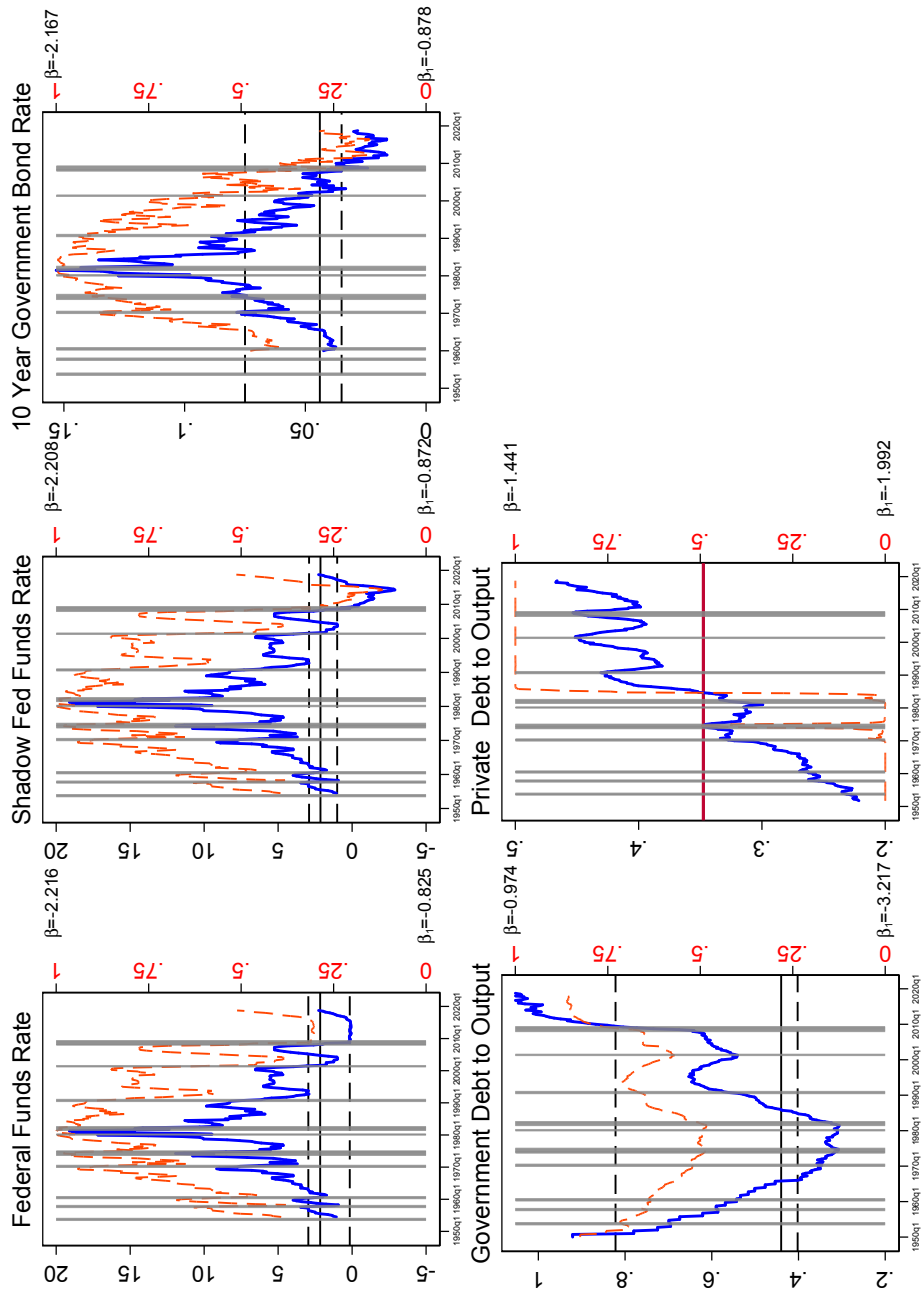


FIGURE C.2: Threshold Variables, Threshold and Transition Probabilities (Structural Variables), NBER dates



Notes: Blue solid lines represent the threshold variable,  $z$ , red dashed lines represent the  $\{0, 1\}$  transition probabilities and the solid black horizontal line represents the threshold,  $\delta$  (with black dashed confidence intervals). The  $\beta$  values on the rhs axis are the Okun parameters in the respective regimes. As before, shaded gray areas represents NBER recession dates at quarterly frequency.

FIGURE C.3: Threshold Variables, Threshold and Transition Probabilities (Policy and Financial), NBER dates



Notes: See notes to Figure C.2.

## C.2 Additional Tables

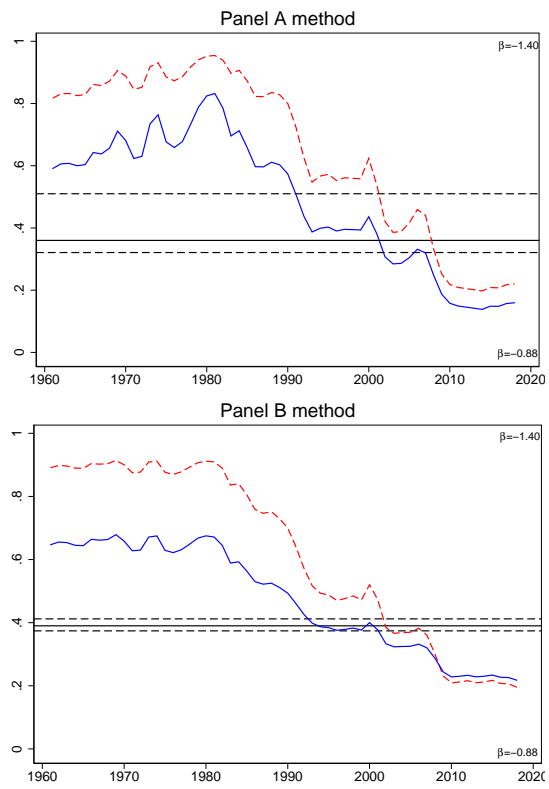
TABLE C.1: Summary Statistics of Thresholds

Variable	Obs	Mean	Median	Std. Dev.	Min	Max	$\delta$
<i>labsh</i>	276	109.294	110.502	4.524	98.150	117.495	107.919
<i>u<sub>l</sub></i>	276	0.161	0.141	0.090	0.033	0.452	0.075
<i>u<sub>n</sub></i>	276	0.055	0.055	0.005	0.046	0.062	0.049
<i>prf</i>	276	0.501	0.537	0.092	0.333	0.601	0.569
<i>d</i>	276	0.581	0.572	0.212	0.306	1.053	0.440
<i>dp</i>	269	0.353	0.356	0.071	0.221	0.467	0.347
<i>i<sub>s</sub></i>	258	4.824	4.560	3.609	0.070	19.100	2.160
<i>i<sub>s,wx</sub></i>	258	4.679	4.560	3.826	-2.890	19.100	2.140
<i>i<sub>l</sub></i>	236	0.061	0.058	0.029	0.016	0.153	0.044
$\sigma_{y_{i(10)}}^c$	237	0.014	0.013	0.004	0.008	0.023	0.020

## D Annual Data Analysis

In this Section of the Appendix, we provide estimates of the model with a single factor of the threshold variables based on annual frequency of the data.

FIGURE D.1: Transition Probability (Annual Frequency of the Data)



Notes: See also notes to [Figure 4](#).

TABLE D.1: Estimates of model (7) with a single factor threshold based on annual data

**Panel A: Estimates based on standardized variables  $\hat{z}_j$**

$\beta_1$	$\beta_2$	$\beta$	$100\omega_1$	$100\omega_2$	$\gamma$	$\delta$	$\mathcal{D}_2$	AIC	$LM^{\text{sup}}$
-1.03 (0.30)	-1.12 (0.45)	-2.15			8.90 [5.100,9.44]	0.36 [0.351,0.371]	69.26	-220.77	
					Ignoring Endogeneity				
-0.88 (0.36)	-1.40 (0.52)	-2.28	0.02 (0.03)	-0.50 (0.25)	6.40 [4.11,8.45]	0.36 [0.321,0.510]	70.50	-218.80	9.30 [0.000]
					Controlling for Endogeneity				

**Panel B: Estimates based on scaled variables  $\hat{z}_j$**

$\beta_1$	$\beta_2$	$\beta$	$100\omega_1$	$100\omega_2$	$\gamma$	$\delta$	$\mathcal{D}_2$	AIC	$LM^{\text{sup}}$
-1.16 (0.25)	-0.95 (0.37)	-2.11			17.70 [7.21,8.425]	0.37 [0.370,0.412]	70.17	-219.93	
					Ignoring Endogeneity				
-0.88 (0.36)	-1.40 (0.52)	-2.28	-0.04 (0.04)	-1.70 (0.87)	8.20 [7.210,8.425]	0.39 [0.374,0.412]	66.67	-219.93	12.80 [0.000]
					Controlling for Endogeneity				

Notes: See notes to Table 5.

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