The Central Bank's Balance Sheet and Treasury Market Disruptions

PRELIMINARY AND INCOMPLETE

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Abstract

This paper presents a dynamic asset pricing model of Treasury bonds with banks subject to both capital and liquidity requirements. Capital requirements and households' preference for money-like assets push non-banks to be the primary holders of Treasuries, thereby exposing Treasury yields to funding shocks originating in repo markets. When holding sufficient reserves, banks mitigate those shocks by lending in repo when the funding supply tightens. When reserves are scarce, banks stop lending. Repo rates and Treasury yields spike up to reflect those funding imbalances. Our model highlights the key role of both sides of the central bank's balance sheet as well as agents' anticipation about the duration of shocks and policy intervention in explaining observed Treasury market disruptions.

Keywords: Repo Markets, Liquidity Risk, Cash-future Basis, Shadow Banks, Balance Sheet Cost, Intraday Liquidity Requirements

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1 Introduction

In recent years, a series of disruptions have affected government debt markets, raising concerns about their stability. Notable episodes include the September 2019 overnight repo rate surge, the March 2020 Treasury market stress, and the September 2022 turmoil in the UK sovereign bond market. These events, characterized by unusual surges in yields, have significant implications for governments' funding capacity and financial stability.

Although a growing body of literature has examined specific market dislocation events,[1] a unified framework able to jointly explain these episodes and present a coherent picture of Treasury market fragility is still lacking. Such a framework is necessary for understanding the fundamental mechanisms behind these market disruptions and studying associated policy-relevant questions.

In this paper, we present a model of Treasury and repo markets that jointly explain these events, emphasizing the role of central banks' balance sheets and dynamic portfolio allocations. In our model, banks are subject to balance sheet cost and intraday liquidity requirements. Households provide repos through banks' dealer subsidiaries to leveraged shadow bank investors, such as hedge funds. Due to the costly nature of banks' balance sheets, shadow banks optimally hold Treasuries financed in repo in equilibrium. Doing so creates liquidity risk for those institutions as a consequence of the combination of Treasury transaction costs and repo supply shocks. When a negative repo supply shock hits, and banks are constrained by their intraday liquidity requirements, shadow banks may prefer to fire-sell Treasuries rather than roll over their positions at a high repo rate.

We use our model to illustrate some of the key mechanisms driving repo and Treasury disruptions. In particular, our model qualitatively matches the yield and portfolio dynamics around the three main sources of repo market instability: (i) quarter-end shocks, (ii) tax-deadline shocks, and (iii) Treasury issuance shocks. Overall, the main novelty of our approach is to consider a model of repo and Treasury markets that is both dynamic—as agents anticipate shocks and policy interventions—and general equilibrium—as every financial asset in the economy is another agent's liability.

Our first contribution is to show that both sides of the central bank's balance sheet are independent key determinants of market disruptions. On the asset side, a large portfolio of Treasury securities held by the central bank diminishes the demand for repo financing from shadow banks, subsequently reducing the likelihood of a rate spike. On the liability side, a large central bank balance sheet with abundant reserves enables banks to utilize those reserves for repo lending when necessary, further decreasing the probability of a spike. Consequently, when

[1] See, for example, He, Nagel, and Song (2022), Avalos, Ehlers, and Eren (2019), Afonso, Cipriani, Copeland, Kovner, La Spada, and Martin (2020), Correa, Du, and Liao (2020), and Copeland, Duffie, and Yang (2022) for studies on the repo market turmoil in September 2019 and March 2020, and Bank of England (2022) for the September 2022 turmoil in the UK sovereign bond market.

the central bank reduces the overall size of its balance sheet, it exerts simultaneous pressure on both the demand and supply for repo financing.

Our second contribution is to relate equilibrium Treasury holdings to the interaction between regulatory frictions and shock expectations. In our model, we find that shadow banks react to a lower probability of a repo spike by allocating a larger portion of their portfolio to Treasuries taking advantage of their low balance sheet cost.[2] This mechanism further implies that a decrease in shock probability leads to an increase in the shock intensity as a result of agents' anticipations. This outcome is reminiscent of the "volatility paradox" described in Brunnermeier and Sannikov (2014), where agents react to an exogenous decrease in risk probability by modifying their portfolios to return to the initial risk level.

The third contribution of this paper is to characterize how the impact of a repo supply shock is distributed between its effects on repo rates and Treasury yields. In our model, since shadow banks are the marginal holders of Treasury securities, which they finance in repo markets, a shock to the cost of repo financing may also affect Treasury yields. This mechanism is present in He, Nagel, and Song (2022) but does not account for why Treasury yields appear more reactive to shocks in some episodes, such as March 2020, while the repo market absorbs the majority of the shock in other episodes, like September 2019. Our model demonstrates that the existence of fixed trading costs implies that the market most affected depends on agents' expectations regarding the shock's duration. Specifically, short-lived shocks will have a greater impact on the repo market than on the Treasury market, as shadow banks are willing to pay high interest rates for a brief period to avoid incurring transaction fixed costs. In contrast, when shadow banks anticipate a long-lasting shock, it becomes less costly to pay the fixed transaction cost rather than a high repo rate over an extended period. This insight supports the observation that repo markets were primarily affected in September 2019 around a temporary tax deadline shock, whereas Treasury market yields were more impacted in March 2020 following the longer-lasting COVID-19 shock.

Our fourth contribution is to consistently qualitatively match empirical observations across various shocks, which allows for a precise characterization of the mechanisms at play. We model intermediation shocks as arising from the contraction of unregulated dealer intermediaries' balance-sheet capacity, as Munyan (2015) documents take place around quarter-ends when foreign dealers window-dress their balance sheets. As observed in the data, a decrease in intermediation capacity leads to intermediation repo spreads, repo yields, and reverse repo take-ups. In addition, as documented by Pozsar (2019) and Correa, Du, and Liao (2020), banks act in our model as a "lender-of-next-to-last-resort" by covering up for the gap in repo supply by draining their reserves down until reaching their reserves constraint. Once this threshold is reached, repo rates hike above the interest on reserves, potentially triggering shadow banks to

[2] This mechanism aligns with the empirical observations from and Avalos et al. (2019) finding increased participation of relative-value hedge funds in Treasury markets, engaging in cash-future basis trades, or essentially providing warehousing for Treasuries while financing themselves overnight in the bilateral repo market.

fire-sell Treasuries.

Tax deadline shocks, representing money market fund outflows around tax deadlines such as the middle of September, have the potential to trigger similar dynamics through a different mechanism. Because tax deadlines are net repo supply shocks, intermediation capacities are not constrained by these shocks, as can be empirically observed by stable intermediation spreads. Rather, tax deadlines simultaneously reduce the supply of repos from households while reducing the quantities of reserves available to banks as those are flowing to Treasury accounts.

A crucial additional insight from the model is that the drawdown of reserves following various shocks can be both helpful and harmful depending on the level of reserves. Until reaching the reserves constraint, a reduction in reserves is beneficial, as it frees up bank balance sheet space, allowing banks to lend in repo the exact amount needed for markets to clear without a surge in rates. Once the reserves constraint is binding, however, a further decrease in reserves becomes problematic as banks' repo supply is restricted by reserves.

We also make use of the model to characterize the set of frictions that is necessary to explain the observed disruptions. We find that repo rates and Treasury yields can only increase beyond the interest on reserves following a repo supply shock with a strict combination of three frictions: banks' balance sheet costs, an intraday reserves requirement, and an active reverse repo facility. In other words, eliminating any of these three frictions would be sufficient to prevent repo rates from spiking upward.

Lastly, the granularity of the framework allows us to study the efficiency of central bank repo facilities in mitigating those frictions. We find that the effectiveness of central bank operations depends on the interplay between the type of repo supply shock and the counterpart involved in the operation. In particular, a repo facility available only to banks but not shadow banks is not effective in addressing repo intermediation shocks. This result occurs because bloated dealer balance sheets are the source of the disruptions under this scenario. As a result, a repo facility proves helpful only if directly accessible to shadow banks, allowing the central bank to effectively act as a substitute intermediary by simultaneously borrowing repos from households at the reverse repo facility and lending to shadow banks at the repo facility. Conversely, we find that repo rate spikes caused by a sharp decline in repo supply due to large corporations paying taxes, such as the event in mid-September 2019, can be effectively mitigated by a repo facility open solely to banks. In this case, banks can utilize their balance sheets to intermediate repos from the central bank to shadow banks, bridging the gap in supply.

Section 2 provides motivating evidence. Section 3 presents a dynamic asset pricing model with financial frictions. The purpose of the model is the study of the mechanisms driving repo and Treasury market disruptions as observed in recent years. To do so, the model makes a minimal set of four assumptions capturing institutional arrangements and frictions existing in most post-GFC financial jurisdictions, including in the US. Section 4 presents the main dynamics of the model. Section 5 verifies that those dynamics correspond to empirical patterns documented in

Section2 and studies the policy implications of the model.

Related Literature Our paper complements the literature highlighting the costly nature of intermediation and the frictions it introduces in Treasury markets. Munyan (2015) and Du, Tepper, and Verdelhan (2018) demonstrate that these frictions manifest as quarter-end effects in repo and FX swap rates, as foreign dealers reduce balance sheet size to comply with leverage ratio regulations. Andersen, Duffie, and Song (2019) reveal the implications of such regulations on funding value adjustments (FVAs) for major dealers, identifying debt overhang costs for shareholders. Correa, Du, and Liao (2020) illustrate how banks engage in "reserves-draining" intermediation to lend in money markets following quarter-end shocks to bypass these constraints. Klingler and Syrstad (2021) provide a comprehensive empirical inquiry across the many factors influencing repo rates. Relevant to our object of study, these constraints have also been found to impact the pricing of Treasury-based arbitrage trades, as seen in cash-future basis (Barth and Kahn, 2023), swap spreads Jermann (2020), and CIP violations Du, Tepper, and Verdelhan (2018). Boyarchenko, Giannone, and Santangelo (2018) further show how dealer balance sheet costs affect repo pricing and arbitrage funding for non-banks. Siriwardane, Sunderam, and Wallen (2022) relate the existence of uncorrelated basis to the ability to use the underlying asset in repo markets and obtain funding to finance the basis trade. He, Nagel, and Song (2022) connect these findings to the extraordinary increase in Treasury yields observed in March 2020 through a preferred habitat model whereby dealers incur an increased cost of holding Treasuries when absorbing fire sales from other sectors. Eisenbach and Phelan (2023) further endogenize Treasury sales in a global game in which investors anticipate dealer balance sheet bottlenecks. Also closely related, Jappelli, Pelizzon, and Subrahmanyam (2023) study the effect of bond scarcity on repo specialness and its implications for the term structure.

In addition, another strand of the literature has pointed to intraday liquidity stress tests and their effect on banks' ability to serve as a stabilizing force in repo markets or "lender-of-next-to-last-resort" and effectively increase the financial system's reliance on reserves. Pozsar (2019) identifies potential liquidity concerns related to Treasury settlements and excess balance sheet normalization. Gagnon and Sack (2019) discuss policy options to address these issues, such as a standing repo facility, higher reserve levels, and explicit directives to control the repo rate. In particular, the repo turmoil of September 2019 has been partially attributed to hedge funds' use of repo to finance Treasury holdings by Avalos, Ehlers, and Eren (2019). Afonso, Cipriani, Copeland, Kovner, La Spada, and Martin (2020) provide a detailed account of the event, highlighting the role of reserves and interbank market frictions, while Anbil, Anderson, and Senyuz (2021) emphasize the role of trading relationships. d'Avernas and Vandeweyer (2022) and Yang (2022) model the impact of intraday liquidity constraints on money market dislocations, finding that non-linearities can generate significant spikes in repo rates. Copeland, Duffie, and Yang (2022) emphasize the role of reserves in alleviating intraday repo payment timing stresses. Acharya and Rajan (2022) and Acharya, Chauhan, Rajan, and Steffen (2023) identify a ratchet effect on banks' liquidity, implying that removing reserves during Quantitative Tightening exposes banks to increased liquidity risk.

Our study distinguishes itself from the cited literature by presenting a comprehensive framework that encompasses both capital and liquidity regulation in a dynamic context, which includes a complete role for the central bank balance sheet. This framework enables us to understand the mechanisms connecting frictions to policy and offers a unique set of implications absent in existing work. In particular, our framework highlights the important role of both sides of the central bank balance as a stabilizing force in Treasury markets, as well as how anticipation of shocks and policy interventions may affect steady-state portfolio choices and, ultimately, the likelihood and magnitude of these shocks.

2 Motivating Evidence

	Repo Rate	Interm. Spread	RRP vol.	TGA vol.
Quarter End	+	+	+	+
Tax Deadline	+	0	−	+
Treasury Issuance	+	+	0	+

Table 1: Qualitative Summary of Empirical Evidence The table provides a qualitative summary of the empirically observed relationship between shocks relevant to repos and Treasury markets: Quarter Ends (reduced intermediation capacity), Tax Deadlines (reduced net repo supply), and Treasury Issuance (increased repo demand) to Repo Rates (measured as the inter-dealer rate, TGCF), Intermediation Spreads (measured as the spread between the inter-dealer rate, TGCF, and the dealer-to-money fund rate, TGCR), Reverse Repo volumes (RRP vol.) at the Fed and Treasury General Account volumes (TGA vol.) at the Fed.

As mentioned previously, a large literature has investigated the dynamics of repo markets empirically. In this section, we motivate our model with a qualitative summary of the relationship between various shocks found to affect repo markets and repo market spreads. Table 1 summarizes the qualitative reaction of repo markets to the main shocks observed in this market. This qualitative table is informed by previous studies and is consistent with the results of a linear regression exercise, which we report separately in the Online Appendix.[3]

As documented by Munyan (2015), Du, Tepper, and Verdelhan (2018), Paddrik, Young, Kahn, McCormick, and Nguyen (2023) and Bassi, Behn, Grill, and Waibel (2023), quarter ends

[3]We choose to relegate these regression tables to the Online Appendix and focus on a qualitative summary due to the highly non-linear nature of the dynamic system we study combined with few observations of large movements, which do not allow for meaningful quantitative estimates of these relationships. For similar reasons, we also abstain from attempting to quantify our model. Rather, we believe that the qualitative heterogeneity in reactions of spreads and volumes to various shocks provides relevant information to discipline a model aimed at understanding the dynamics of Treasury and repo markets.

represent a major source of repo market disruptions as foreign dealer intermediation contracts on those days following window dressing practices. Contrarily to the US, most foreign jurisdictions are calculating the Basel III-mandated bank leverage ratio based on a snapshot of banks' balance sheets at quarter ends rather than an average over all days of the quarter, as is the practice in the US. Foreign dealer banks tend to contract their balance sheets and cut back on low-yielding activities such as repo intermediation on those days to account for increased Funding Value Adjustments (Andersen, Duffie, and Song, 2019). Accordingly, the first row of 1 shows an increase in repo rates relative to the interest on reserves on those dates, as well as an increase in intermediation spreads[4], an increase in the Fed's reverse repo facility usage (RRP) as well as an increase in the Treasury General Account (TGA) at the Fed.

In addition, several works (Copeland, Duffie, and Yang, 2022; d'Avernas and Vandeweyer, 2021) also point to tax deadlines being an important driver of repo market disruption. In particular, as noted by Anbil, Anderson, and Senyuz (2021), the largest spike in repo rates on record took place on September 16 and 17, 2019, corresponding to a corporate tax deadline. On those dates, corporations tend to withdraw cash from their money market accounts to the Treasury General Account. These flows generate both a net decrease in repo supply and a decrease in reserves available to banks. The third row of 1 correspondingly shows larger repo rates along tax deadlines (column 1) that are not accompanied by a significant increase in intermediation spread (column 2) or in volumes at the reverse repo facility (column 3). As expected, the Treasury General Account increases on tax deadline days (column 4).

Lastly, Treasury issuance has also been shown to be a factor of increased tension in repo markets (Klingler and Syrstad, 2021; Paddrik, Young, Kahn, McCormick, and Nguyen, 2023). As the Treasury issues additional debt securities, securities dealers and asset managers tend to purchase and finance those purchases in the repo market, thereby increasing the demand for repo. As can be seen in the fifth row of 1, issuance of Treasuries is associated with increased repo rates (column 1), repo intermediation spreads (column 2), and larger balance in the Treasury General Account (column 3).

In the following sections, we develop a model of Treasury and repo markets that allows us to explicitly incorporate those shocks and study the mechanisms underscoring the pattern in prices and volumes documented here.

[4]Intermediation spreads are calculated as in He, Nagel, and Song (2022) as the difference between $TGCF$, an index of inter-dealer repo rates representing the rate at which dealers are willing to lend, and $TGCR$, an index of repo rates based on dealer-to-money-fund transactions, corresponding to the rate at which dealers are willing to borrow. Hence, this difference is considered a good indicator for repo intermediation margins.

3 Model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space that satisfies the usual conditions and assume that all stochastic processes are adapted. The economy evolves in continuous time with $t \in [0, \infty)$ and is populated by a continuum of traditional banks (with a dealer subsidiary), shadow banks, and households, as well as the Treasury and a central bank. Figure 1 depicts the balance sheets of the different sectors in the economy. The Treasury issues Treasury bonds against future tax liabilities and maintains a balance in the Treasury General Account (TGA); the central bank holds outstanding Treasury bonds and issues reserves to the banking sector; and households invest their wealth and future tax in repo and deposits. Traditional banks hold securities, reserves, and some Treasury bonds leveraged by issuing deposits. Traditional banks can also either lend or borrow in repo. Shadow banks hold Treasury bonds financed with repo.

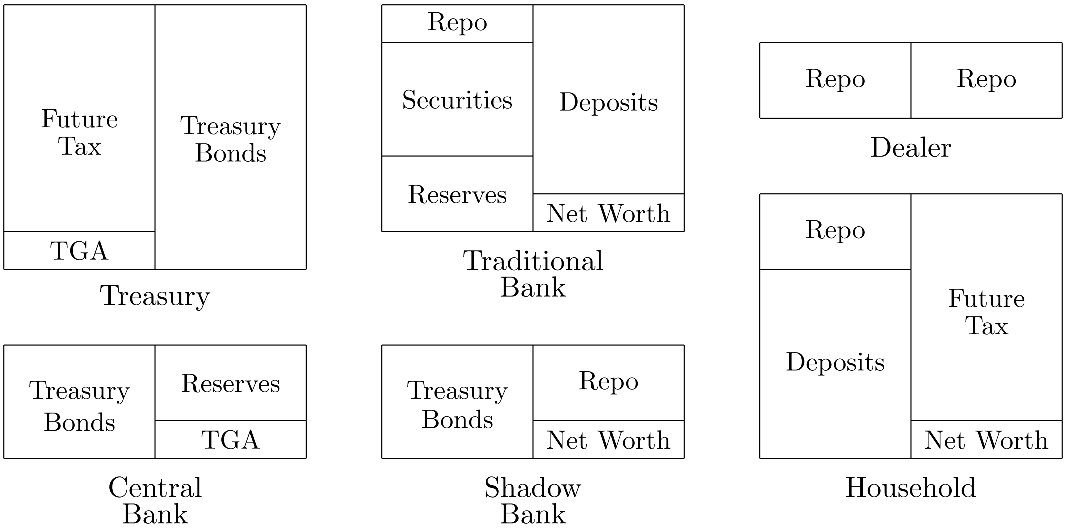

Figure 1: Chart of Sectors' Balance Sheets

Main Frictions Four economic forces and frictions play an important role in our framework. First, households, which includes firms investing in money market funds, have preferences regarding the composition of their portfolio of liquid assets, such as repos and traditional bank deposits. This assumption captures that money market fund shares and bank deposits have different properties in terms of safety, yields, and liquidity, making those imperfect substitutes for households and corporate treasury. Second, traditional banks and their dealer subsidiaries are subject to a balance sheet size cost, capturing the debt-overhang cost of equity issuance in the presence of leverage ratio regulation(Andersen, Duffie, and Song, 2019). Thirdly, Intraday Liquidity (IL) regulations mandate that traditional banks maintain a buffer of reserves at all times during the day, which limits their ability to lend in repo markets to shadow banks (d'Avernas and Vandeweyer, 2022). Fourth, buying or selling Treasury securities incurs a transaction cost. Although the US Treasury market is arguably one of the most liquid markets in the world, trading Treasury still entails paying for non-trivial bid-ask spreads, leading to significant. Our analysis aims at understanding how these forces interact together to explain recent events in Treasury and repo markets.

3.1 Environment

Preferences Bankers have logarithmic preferences over their consumption rate c_t of their net worth n_t with a time preference ρ:

$$\mathbb{E}_t \left[\int_t^\infty e^{-\rho(u-t)} \log(c_u n_u) du \right]. \tag{1}$$

Households further value liquidity services provided by holding repo and deposits,

$$\mathbb{E}_t \left[\int_t^\infty e^{-\rho(u-t)} \Big(\log(c_u n_u) + \beta \log(h(w_u^p, w_u^d; \alpha_u) n_u) \Big) du \right], \tag{2}$$

where h is a Cobb-Douglas aggregator of deposits and repo portfolio weights w_t^d and w_t^h,

$$h(w_t^p, w_t^d; \alpha_t) = (w_t^d)^{\alpha_t} (w_t^p)^{1-\alpha_t}. \tag{3}$$

The parameter α_t corresponds to the preference of households for holding repo relative to deposits.

Technology There is a unit of risk-free capital producing a flow of real output, y, with constant productivity. We assume that capital can only be held by traditional banks to instead focus our analysis on the transactions of Treasuries, repos, and reserves.

Treasury The Treasury issues bonds against the future tax liabilities of households and is responsible for administrating redistributive lump-sum tax policies. The net present value of future tax liabilities must equal the outstanding amount of Treasuries: $\tau_t^h n_t^h + a = b_t$, where b_t is the quantity of bonds issued, a_t is the size of the TGA account, and τ_t^h is the future tax liability of households per unit of wealth. Although Treasury bonds are long-lived assets, we assume those have an instantaneous return because we are interested in the liquidity risk of shadow banks rather than their interest rate risk exposures. A straightforward interpretation of a Treasury holding in our model is the combination of a Treasury long position along with a futures contract selling that Treasury in the future and thereby hedging its duration risk. Similarly, the spread between the Treasury yield and the repo rate is to be interpreted as the cash-future basis (see Barth and Kahn (2023) for an in-depth account of this trade by hedge funds in the post-Basel III environment).

Central Bank The central bank holds Treasury bonds, $\underline{b}$, financed by reserves held by banks m_t and in the Treasury General Account (TGA) a_t. The underline notation differentiates the central bank's holdings of Treasury bonds $\underline{b}_t$ from the bonds issued by the Treasury b. The central bank can also lend repo at the repo facility, rp_t, and borrow repo at the reverse repo facility, rrp_t. Thus, the balance sheet constraint for the central bank is given by

$$\underline{b}_t + rp_t = m_t + a_t + rrp_t. \tag{4}$$

The central bank determines the interest rates at which the central bank lends at the repo facility and at which it borrows at the reverse repo facility. These rates are denoted by $r^{rp} > r^m$ and $r^{rrp} < r^m$, respectively. Thus, in the presence of facilities, it is the net quantities lent $rp_t - rrp_t$ that determine the quantity of reserves in the banking sector m_t as a residual. In addition, for simplicity, we assume that the central bank always operates with zero net worth and instantaneously transfers all seigniorage revenues to the Treasury.

Dealers We make the assumption that the repo market is fully intermediated, meaning that households exclusively invest in repos through bank dealer subsidiaries rather than directly with traditional or shadow banks.[5] In order to account for fluctuations in intermediation capacities, our model includes a foreign dealer possessing a balance sheet size denoted by f_t. This foreign dealer operates alongside the traditional bank dealer subsidiary to intermediate repos between households and banks. Variations in the foreign dealer's balance sheet size can be thought of as a representation of the window-dressing practices observed at quarter ends among foreign dealers, as described by Munyan (2015). Doing so allows for a transparent mapping between the studied empirical regularities and the model. Although we motivate the model mostly from these events, a repo intermediation shock can also be interpreted as any event that negatively affects aggregate dealer balance sheet capacity.[6]

Shocks The liquidity preference parameters α_t, intermediation by foreign dealers f_t, the TGA account a_t, Treasuries b_t, and the central bank's balance sheet $\underline{b}_t$ are subject to aggregate shocks. The vector of time-varying parameters, denoted by $\mathbf{x}_t \equiv \{\alpha_t, f_t, a_t, \underline{b}_t, b_t\}$, follows

$$d\mathbf{x}_t = \begin{cases} (\mathbf{x}^s - \mathbf{x}_t)dN_t & \text{if} \quad \mathbf{x}_t \neq \mathbf{x}^s, \\ (\mathbf{x}' - \mathbf{x}_t)dN_t & \text{if} \quad \mathbf{x}_t = \mathbf{x}^s, \end{cases} \tag{5}$$

[5] This assumption aligns with the actual institutional framework in the US, where the vast majority of repo transactions are effectively intermediated by securities dealers. For further institutional details on repo markets, we refer to the work of Copeland, Martin, and Walker (2014).

[6] For instance, the increasingly popular practice of FICC-sponsored cleared repo, which allows netting for regulatory purposes would be interpreted in our setting as equivalent to an increase in the foreign dealer repo capacity.

where dN_t is a Poisson process with intensity $\lambda_t > 0$ and $\mathbf{x}'$ is a random variable independent and identically distributed. We specify the exact distribution of $\mathbf{x}'$ in the following sections when we study shocks to different parameters. As shown in equation (5), the economy features a steady state $\mathbf{x}^s$. Upon the arrival of a Poisson shock in state $\mathbf{x}_t = \mathbf{x}^s$, the parameters $\mathbf{x}_t$ takes on a new random value $\mathbf{x}'$. Upon the arrival of a Poisson shock in state $\mathbf{x}_t \neq \mathbf{x}^s$, $\mathbf{x}_t$ reverts to $\mathbf{x}^s$. The Poisson intensity λ_t is equal to λ if $\mathbf{x}_t = \mathbf{x}^s$ and equal to λ' otherwise. Hence, λ represents the likelihood of an aggregate shock while λ' determines the expected duration of the shock. All agents have perfect information on the Poisson process.

Overlapping Generations In order to abstract from sectorial wealth dynamics, we assume that agents are short-lived in an overlapping generations framework such that the wealth shares of the traditional bank sector, shadow bank sector, and household sector—respectively given by $n_t/N_t, \overline{n}_t/N_t$, and n_t^h/N_t, where $N_t = n_t + \overline{n}_t + n_t^h$—are constant over time.[7]

3.2 Agent Problems

Traditional Banks Traditional banks face a Merton's (1969) portfolio choice problem augmented with transaction costs and a balance sheet cost. Traditional banks maximize their lifetime expected logarithmic utility:

$$\max_{\{c_u \geq 0, w_u^k \geq 0, w_u^b \geq 0, w_u^m \geq 0, w_u^p, w_u^x \geq 0, w_u^d \geq 0\}_{u=t}^{\infty}} \mathbb{E}_t \left[\int_t^{\infty} e^{-\rho(u-t)} \log(c_u n_u) du \right], \tag{6}$$

subject to the law of motion of wealth:

$$\begin{aligned} dn_t = & \left(w_t^k r_t^k + w_t^b r_t^b + w_t^m r_t^m + w_t^p r_t^p + w_t^x (r_t^p - r_t^{pt}) - w_t^d r_t^d - c_t \right) n_t dt \\ & - \frac{\chi}{2} \ell_t^2 n_t dt + (e^{-\nu(|dw_t^b|)} - 1) n_t, \end{aligned} \tag{7}$$

the balance sheet constraint:

$$w_t^k + w_t^b + w_t^m + w_t^p = 1 + w_t^d, \tag{8}$$

and the IL constraint:

$$w_t^p \leq \kappa w_t^m. \tag{9}$$

Traditional bankers choose their consumption rate c_t, their portfolio weights for capital w_t^k, Treasury bonds w_t^b, reserves w_t^m, and deposits w_t^d given their respective interest rates r_t^k, r_t^b, r_t^m, and r_t^d. Traditional banks choose the portfolio weight for repo w_t^p given the interest rates in the bilateral repo market r_t^p (to shadow banks). Traditional banks select the size of their dealer

[7] Thus, ρ is the effective time discount rate net of the death rate.

balance sheet w_t^x to profit from the spread between the bilateral and triparty repo rates (from households) but incur a balance sheet cost. The balance sheet cost is quadratic in bank leverage ℓ_t with a cost parameter χ where

$$\ell_t \equiv w_t^d - \min\{0, w_t^p\} + w_t^x. \tag{10}$$

Banks are subject to transaction costs when trading Treasury bonds. The wealth after a transaction n_t is equal to

$$n_t = n_{t\text{-}} e^{-\nu|dw_t^b|}, \tag{11}$$

where ν is the transaction cost and t- is the time prior to the transaction.[8,9,10]

Finally, traditional banks are subject to an intraday liquidity (IL) constraint, which limits repo lending to a fraction κ of reserve holdings. This constraint stems from new regulations such as resolution liquidity execution need (RLEN) penalizing banks for using the Feds' intraday overdraft and, thereby, making the quantity of reserves available for intraday settlements and repo clearing the binding constraint. We refer to a companion paper d'Avernas and Vandeweyer (2022) for further details on how this intraday liquidity differs from the traditional bank reserve requirement (revoked in the US since February 2023), which was applied to overnight balances averaged over a 14-day maintenance period.

Shadow Banks Shadow banks face the same problem as traditional banks but without balance sheet costs. Shadow banks maximize their lifetime expected logarithmic utility:

$$\max_{\{\bar{c}_u \geq 0, \bar{w}_u^b \geq 0, \bar{w}_u^p\}_{u=t}^{\infty}} \left[\int_t^{\infty} e^{-\rho(u-t)} \log(\bar{c}_u \bar{n}_u) du \right], \tag{12}$$

subject to the law of motion of wealth:

$$d\bar{n}_t = \left(\bar{w}_t^b r_t^b - \bar{w}_t^p r_t^p - \bar{c}_t \right) \bar{n}_t dt + (e^{-\nu|d\bar{w}_t^b|} - 1)\bar{n}_t, \tag{13}$$

and the balance sheet constraint:

$$\bar{w}_t^b = 1 + \bar{w}_t^p. \tag{14}$$

[8] Thus, $dw_t^b = w_t^b - w_{t\text{-}}^b$.

[9] We abstract from potential transaction costs on capital as capital can only be held by traditional banks and focus our analysis on the transaction of Treasury bonds. Allowing only banks to hold capital is without loss of generality as long as the transaction cost on capital is larger than the transaction cost on Treasury bonds.

[10] We abstract from the impact of transaction costs on aggregate consumption—that is, the magnitude of these costs is too small to have a general equilibrium impact. Thus, the market clear condition for consumption is simply given by $C_t = y$ where C_t is aggregate consumption.

Shadow banks choose holdings of Treasury bonds $\overline{w}_t^b$ and repo financing $\overline{w}_t^p$ given the respective interest rates r_t^b and r_t^p. As traditional banks, shadow banks also incur a similar transaction cost when purchasing or selling Treasury bonds.

Households Households maximize lifetime utility of consumption and liquidity benefits:

$$\max_{\{c_u^h \geq 0, w_u^{h,d} \geq 0, w_u^{h,p} \geq 0\}_{u=t}^{\infty}} \mathbb{E}_t \left[\int_t^{\infty} e^{-\rho(u-t)} \left(\log(c_u^h n_u^h) + \beta \log(h(w_u^{h,p}, w_u^{h,d}, \alpha_u)) \right) du \right], \tag{15}$$

subject to the law of motion of wealth:

$$dn_t^h = \left(w_t^{h,p} r_t^{p,t} + w_t^{h,d} r_t^d - c_t^h - r_t^{\tau} \right) n_t^h dt, \tag{16}$$

and the balance sheet constraint:

$$w_t^{h,p} + w_t^{h,d} = 1 + \tau_t^h. \tag{17}$$

Households choose consumption c_t^h and their portfolio holdings of repo $w_t^{h,p}$ and $w_t^{h,d}$ given their liquidity preference α_t and pay lump-sum taxes $r_t^{\tau} n_t^h$.

Treasury Budget Constraint The budget constraint for the Treasury is given by

$$r_t^b b_t = r_t^{\tau} n_t^h + r_t^b \underline{b}_t + r_t^{rp} rp_t - r_t^m m_t - r_t^{rrp} rrp_t. \tag{18}$$

To pay interest on Treasury bonds, the Treasury collects taxes from households and seigniorage revenues rebated from the central bank.

3.3 Solving

We provide a definition for a Markov equilibrium, make further assumptions to restrict the set of equilibria to focus on empirically relevant scenarios, and derive first-order conditions. We assume that central bank policies $\{r^{rp}, r^{rrp}\}$ are constant over time. Given the assumptions on the aggregate shocks, the state space of the economy is given by the vector of time-varying parameters $\mathbf{x}_t$.[11]

Definition 1. *Given central bank policies $\{r^{rp}, r^{rrp}\}$, a* ***Markov equilibrium*** *$\mathcal{M}$ in $\mathbf{x}_t$ is a set of functions $g_t = g(\mathbf{x}_t)$ for (i) interest rates $\{r_t^k, r_t^b, r_t^m, r_t^p, r_t^{tp}, r_t^d\}$; (ii) individual con-*

[11]The aggregate Treasury portfolio weights are not state variables because, with logarithmic utility and the functional form for transaction costs defined in equation (11), the marginal utility cost of transactions is constant and not a function of the size of the transaction. Thus, if it is profitable to fire-sell Treasuries when entering in state $\{\mathbf{x}_t, w_{t\text{-}}^b, \overline{w}_{t\text{-}}^b\}$, it is also profitable to fire-sell Treasuries when entering in state $\{\mathbf{x}_t, w_{t\text{-}}^{b\prime}, \overline{w}_{t\text{-}}^{b\prime}\}$, for all $w_{t\text{-}}^{b\prime}$ and $w_{t\text{-}}^{b\prime}$. Therefore, the Markov functions are such that $g(\mathbf{x}_t, w_{t\text{-}}^{b\prime}, \overline{w}_{t\text{-}}^{b\prime}) = g(\mathbf{x}_t)$ and the equilibrium is entirely determined by $\mathbf{x}_t$.

trols for traditional banks $\{w_t^k, w_t^b, w_t^m, w_t^p, w_t^x, w_t^d, c_t\}$*, shadow banks* $\{\overline{w}_t^b, \overline{w}_t^p, \overline{c}_t\}$*, and households* $\{w_t^{h,p}, w_t^{h,d}, c_t^h\}$ *such that:*

1. *Agents' optimal controls (ii) solve their respective problems given prices (i).*
2. *The balance sheet constraint of the central bank is satisfied.*
3. *The balance sheet constraint and budget constraint of the Treasury are satisfied.*
4. *Markets clear:*

(a) output:	$c_t n_t + \overline{c}_t \overline{n}_t + c_t^h n_t^h = y,$
(b) capital:	$w_t^k n_t = 1,$
(c) Treasury bonds:	$\overline{w}_t^b \overline{n}_t + w_t^b n_t = b - \underline{b}_t,$
(d) reserves:	$w_t^m n_t = m_t,$
(e) triparty repo:	$w_t^{h,p} n_t^h = w_t^x n_t + rrp_t + f_t,$
(f) bilateral repo:	$(w_t^x + w_t^p) n_t + rp_t + f_t = \overline{w}_t^p \overline{n}_t,$
(g) deposits:	$w_t^d n_t = w_t^{h,d} n_t^h.$

5. *The law of motion for* x_t *is consistent with agents' perceptions.*

Equilibrium Restrictions To simplify the exposition and derive analytical results, we focus our analysis on a subset of equilibria, thereby implicitly restricting the set of parameters and central bank and Treasury policies considered.

First, we focus on equilibria in which traditional and shadow banks are leveraged: $\ell_t > 0$ and $\overline{w}_t^p > 0$; the traditional bank dealer subsidiary has a positive balance sheet size: $w_t^x > 0$; and reserves are in strict positive supply: $m_t > 0$. Furthermore, in the steady state ($\mathbf{x}_t = \mathbf{x}^s$), traditional banks do not invest in repo: $w^p(\mathbf{x}^s) = 0$; hold some treasuries: $w^b(\mathbf{x}^s) > 0$; and the IL constraint is not binding. Finally, the household balance sheet must be larger than the quantity of repo available from traditional banks: $(1 + \tau_t^h) n_t^h > \kappa m_t$; and there is always a state in which paying the transaction cost is optimal: $\exists \mathbf{x}' : w^b(\mathbf{x}') \neq w^b(\mathbf{x}^s)$.

First-order Conditions Applying the maximum principle, we derive the first-order conditions for all agents. With logarithmic preferences, every agent always consumes a fixed proportion of their wealth ρ: $c_t = \overline{c}_t = c_t^h = \rho$. Thus, the market clearing condition for consumption is such that aggregate wealth is constant over time: $N_t = y/\rho$.

The first-order conditions for reserves, bilateral repo, and triparty repo of traditional banks are given by

$$r_t^k - r_t^d = \chi \ell_t, \tag{19}$$

$$r_t^k - r_t^m = \kappa \vartheta^m, \tag{20}$$

$$r_t^k - r_t^p \begin{cases} = -\vartheta^m & \text{if } w_t^p > 0, \\ \in [-\vartheta^m, \chi \ell_t] & \text{if } w_t^p = 0, \\ = \chi \ell_t & \text{if } w_t^p < 0, \end{cases} \tag{21}$$

$$r_t^p - r_t^{pt} = \chi \ell_t, \tag{22}$$

where ϑ^m is the shadow price of the IL constraint. In equation (19), traditional banks equalize the marginal benefits of issuing deposits (return on capital) to its marginal cost (the marginal increase in the balance sheet cost). In equation (20), the marginal cost of holding a unit of reserves must equate to the marginal benefit of loosening the IL constraint. Similarly, in equation (21), if a traditional bank invests in repos, the bilateral repo rate needs to compensate for the tightening of the IL constraint. If a traditional bank funds itself in repo, the repo rate must be sufficiently low to compensate for the increase in the balance sheet cost, similar to deposits. Finally, in equation (22), traditional banks require a spread between bilateral and triparty repo to compensate for the balance sheet cost incurred by intermediating repo at the dealer subsidiary.

Next, the households' first-order condition for their relative holdings of triparty repo and deposit is given by

$$r_t^{pt} - r_t^d = \rho \frac{\beta}{1+\beta} \left(\frac{\alpha_t}{w_t^{h,d}} - \frac{1-\alpha_t}{w_t^{h,p}} \right). \tag{23}$$

Households equalize the marginal benefit of investing in triparty repo over deposits, as given by the spread between the rates on these two assets, to the marginal convenience cost of reallocating one unit of wealth from deposits to repo in the right-hand side of equation (23).

Lastly, the optimality conditions for Treasury portfolio weights are then given by

$$\nu \operatorname{sign}(w_t^b - w_{t\text{-}}^b) n_t V_n(n_t, w_t^b; \mathbf{x}_t) = V_w(n_t, w_t^b; \mathbf{x}_t), \tag{24}$$

$$\nu \operatorname{sign}(\overline{w}_t^b - \overline{w}_{t\text{-}}^b) \overline{n}_t \overline{V}_n(\overline{n}_t, \overline{w}_t^b; \mathbf{x}_t) = \overline{V}_w(\overline{n}_t, \overline{w}_t^b; \mathbf{x}_t), \tag{25}$$

where $n_t = n_{t\text{-}} e^{-\nu |w_t^b - w_{t\text{-}}^b|}$ and $\overline{n}_t = \overline{n}_{t\text{-}} e^{-\nu |\overline{w}_t^b - \overline{w}_{t\text{-}}^b|}$ and $V(n, w; \mathbf{x})$ and $\overline{V}(n, w; \mathbf{x})$ are the value functions of traditional and shadow banks, respectively. Both traditional and shadow banks trade off the marginal cost of a transaction, on the left-hand side, against the marginal benefit of purchasing or selling Treasury bonds, on the right-hand side. This decision depends on current and future rates and the stochastic process for the state variables as captured by the partial derivatives of the value functions.

State Space Partitioning We define five disjoint sets of equilibria corresponding to different dynamics in the pricing of the bilateral repo. Our analysis below characterizes how shocks in state variables $\mathbf{x}$ are shifting equilibrium across those sets.

Definition 2. *Let $\mathcal{A}$ be the set of* **arbitraged** *repo market equilibria, defined as* $\{\mathcal{M}(\mathbf{x}) \in \mathcal{A} \mid w^p(\mathbf{x}) < 0\}$.

Definition 3. *Let $\mathcal{S}$ be the set of* **segmented** *repo market states, defined as* $\{\mathcal{M}(\mathbf{x}) \in \mathcal{S} \mid w_t^p = 0$ *and* $r^p(\mathbf{x}) < r^m(\mathbf{x})\}$.

Definition 4. *Let $\mathcal{U}$ be the set of* **unconstrained** *repo market states, defined as* $\{\mathcal{M}(\mathbf{x}) \in \mathcal{U} \mid r^p(\mathbf{x}) = r^m(\mathbf{x})\}$.

Definition 5. *Let $\mathcal{C}$ be the set of* **constrained** *repo market states, defined as* $\{\mathcal{M}(\mathbf{x}) \in \mathcal{C} \mid r^p(\mathbf{x}) > r^m(\mathbf{x})\}$.

Definition 6. *Let $\mathcal{F}$ be the set of* **firesale** *states, a subset of constrained repo market states $\mathcal{C}$, defined as* $\{\mathcal{M}(\mathbf{x}) \in \mathcal{F} \mid \overline{w}^b(\mathbf{x}) < \overline{w}^b(\mathbf{x}^s)\}$.

We define as *arbitraged*, the set of equilibria in which traditional banks are borrowing in bilateral repos, and as *segmented* the ones in which traditional banks are not marginal in bilateral repos due to the balance sheet cost. As will be clear below, these equilibria arise in cases in which the preference for repo is high relative to deposits. We also define as *unconstrained* equilibria in which traditional banks are net lenders of repos, and the IL constraint is not binding, corresponding to bilateral repo rates being equal to interest on reserves. We define as *constrained* equilibria in which traditional banks are constrained by the IL regulation. In this section of the state space, bank repo rates are above the interest on reserves. Lastly, we define as *firesale*, the set of equilibria in which shadow banks fire-sell Treasury bonds to traditional banks when entering into the shock state. All proofs of lemmas and propositions in the following sections are relegated to Appendix A.

4 Dynamics of a Repo Demand Shock

To gain intuition about model dynamics, we first focus in this section on a shock to the liquidity preference parameter α_t and study the impact of a shortage of repo funding on Treasury markets. That is, we assume that α_t is the only parameter that varies after an aggregate shock—that is, $\mathbf{x}^s = \{\alpha_t, f^s, a^s, \underline{b}^s, b^s\}$ and $\mathbf{x}' = \{\alpha', f^s, a^s, \underline{b}^s, b^s\}$ where α' is independent and identically distributed according to a uniform distribution on $(\alpha^s, 1)$. When α increases, households would like to reduce their holdings of repo relative to deposits, leading to either traditional banks covering the resulting funding gap of shadow banks by providing more repo or an increase in the spreads between repo and deposits to incentives households to retain their original portfolio despite the change in preferences. For simplicity, we also further assume that the repo and

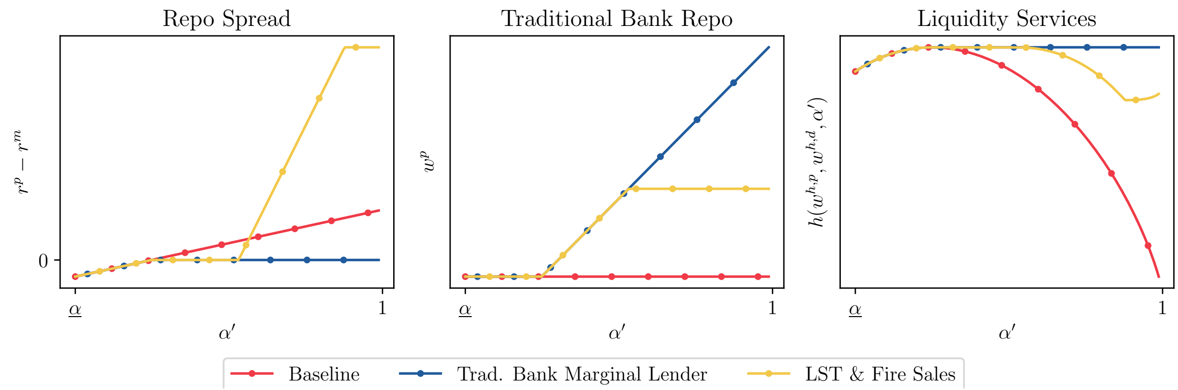

Figure 2: Repo Markets with Fully Rigid and IL-Constrained Bank Balance Sheets. The Figure provides a graphical representation of repo spreads, traditional banks repo lending, and households liquidity services as a function of the shock parameter α' for a numerical example of a given equilibrium under three different restrictions: (1) in red is a baseline case with no feasible repo lending or Treasury sale, (2) in blue is a model allowing banks to lend in repos without an IL constraint (3) in yellow introduces the IL constraint and a finite transaction cost on Treasuries.

reverse repo facilities are closed and relegate the study of these policy instruments to the next sections.

Equilibrium Treasury Holdings We first provide some insights on how steady-state equilibrium allocations are influenced by our four frictions. Our first lemma relates shadow bank equilibrium Treasury positions to banks' balance sheet costs.

Lemma 1. *A larger balance sheet cost parameter, χ, corresponds to a larger shadow banks' Treasury portfolio allocation, $\overline{w}_t^b \overline{n}$ in the steady state. That is,*

$$\frac{\partial (w^b(\alpha^s)\overline{n})}{\partial \chi} > 0.$$

This lemma shows an unintended consequence of regulation increasing banks' balance sheet costs; it pushes liquidity transformation into the unregulated shadow bank sector. Some of the intuition behind this result can be observed by combining equations (8), market clearing condition *(c)*, and equation (10) to write equilibrium bank leverage as:

$$\ell(\alpha) = 1/n(\alpha) + m/n(\alpha) + (b - \underline{b} - \overline{n}(\alpha)\overline{w}^b)/n(\alpha) - 1 + w^x(\alpha).$$

From this expression, we see that bank leverage is decreasing in shadow banks' Treasury holdings. In other words, shadow bank Treasury holdings economize on aggregate banks' costly balance sheet space. As a consequence, a larger balance sheet cost parameter increases this comparative advantage of shadow banks in holding Treasuries, yielding larger shadow bank holdings in equilibrium.

Rigid Balance Sheet Benchmark Figure section 4 illustrates the adjustments in repo spread $r^p - r^m$, the quantity of repo lending by traditional banks w^p, and the value of liquidity services h as a function of the shocked state α'. The red lines represent a benchmark case where traditional banks cannot adjust their quantity of repos, necessitating rate adjustments to compensate households for maintaining the fixed composition of repos and deposits. The blue line scenario allows banks to become marginal lenders in the repo market without being subject to the IL constraint. When traditional bank repo lending is not constrained by their intraday requirements, the economy responds to the negative repo preference shock by having banks borrow more in deposits from households and intermediate those funds to shadow banks through repo, thus optimizing the composition of households' portfolios. However, once the IL constraint becomes binding, banks can no longer lend in repos, and the equilibrium requires households to absorb a suboptimal mix of repos and deposits, leading to increased repo spreads. For a finite transaction cost, repo spreads can increase up to a point where it becomes more profitable for shadow banks to pay the transaction cost and sell Treasuries to traditional banks to reduce their balance sheet rather than paying the prohibitive repo rate. This Treasury fire sales provides an outside option for shadow banks and caps the level at which repo rates can move. We discuss below the dynamics of this fire-sale decision, which on agents' portfolio decisions in anticipation of equilibrium shock frequency, intensity, and duration.

Paradox of Prudence in Repo Markets In our model, the intensity of repo and Treasury yield movements is endogenous and depends on ex-ante portfolio allocations. We find this feedback loop connects the frequency of the shock to its intensity. Proposition proposition 1 formalizes this insight.

Proposition 1. *Lower shock frequency λ results in a higher probability of a repo spike and larger expected Treasury yield spikes, conditional on the arrival of a repo supply shock:*

$$\frac{\partial \mathbb{P}[\alpha' \in \mathcal{F}]}{\partial \lambda} < 0 \quad \textit{and} \quad \frac{\partial \mathbb{E}[r^p(\alpha') - r^m(\alpha^s)]}{\partial \lambda} < 0. \tag{26}$$

Furthermore, lower shock frequency λ incentivizes shadow banks to take more leverage in the steady state:

$$\frac{\partial \overline{w}^b(\alpha^s)}{\partial \lambda} < 0. \tag{27}$$

Proposition 1 demonstrates that when the frequency of shocks decreases, shadow banks optimally react by increasing their leveraged Treasury holdings in the steady state, which eventually results in a larger expected Treasury yield surge once the Poisson shock hits. This result is akin to the "Paradox of Prudence" in Brunnermeier and Sannikov (2014) with the additional feature that agents are trading off risk frequency for risk intensity. These findings are illustrated in Figure 3.

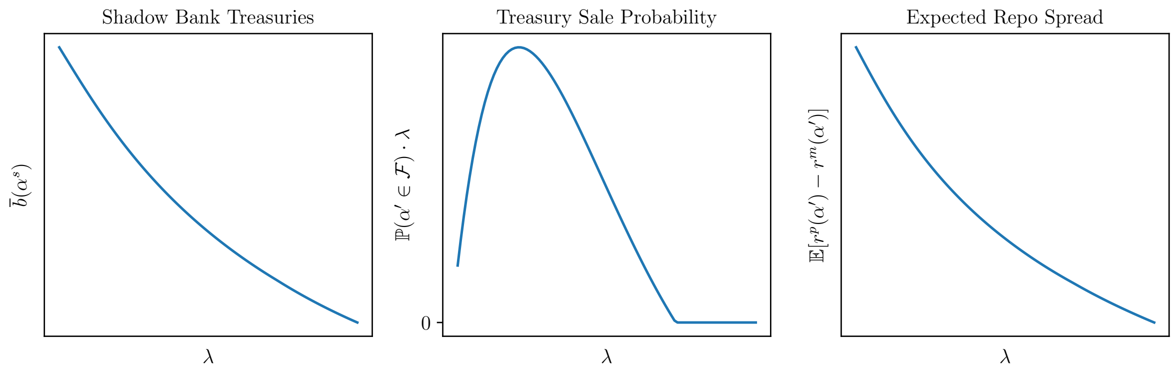

Figure 3: Simple Dynamic Model: Repo Supply Shock Frequency. Repo supply shock frequency is denoted by λ, the intensity of the Poisson process of shocks from the normal state. As supply shock frequency decreases, the risk for rising repo rates due to lower repo supply decreases, to which shadow banks respond by increasing Treasury holdings in the normal state. These increased shadow bank Treasury bond holdings lead to higher probabilities for fire sales and high expected Treasury yield spikes.

Shock Duration, Repo Rates, and Treasury Yields We next consider how expectations about the duration of the shock affects equilibrium prices upon entering in the shock state. In particular, Proposition 2 shows that short-lived shocks result in high repo rate spikes and low Treasury yield surges in expectation, while long-lived shocks result in high Treasury yield spikes and low repo spikes. When a repo supply shock is expected to be short-lived, shadow banks are willing to pay a high repo rate for a short period of time to avoid paying costly round-trip transaction fees, which reduces the likelihood of a fire sale. In contrast, if a shock is expected to last for a long period of time, shadow banks will prefer to sell Treasuries rather than having to pay high repo rates for a potentially long period of time. Shadow banks reducing Treasury bond holdings reduces repo demand, allowing repo rates to decline in the shocked states. This asymmetry might help elucidate why repo rates experienced a dramatic spike in September 2019 while Treasury yields remained relatively stable, as opposed to the events of 2020 when Treasury yields rose sharply, but repo rates did not surge significantly. The September 2019 spike was likely transitory and attributable to the tax deadline, while the March 2020 shock occurred amid the COVID-19 pandemic, characterized by considerably greater uncertainty regarding its duration and long-term impact.

Proposition 2. *The expected duration of the repo supply shock affects equilibrium. In particular, shorter shock duration leads to a reduced probability of fire sales, conditional on Poisson supply shock arrival:*

$$\frac{\partial}{\partial \lambda'} \mathcal{P}(\alpha' \in \mathcal{F}) < 0. \tag{28}$$

Furthermore, shorter shock duration leads to lower expected Treasury yield spikes and higher *expected repo spikes:*

$$\frac{\partial}{\partial \lambda'} \mathbb{E}[r^b(\alpha')] - r^b(\alpha^s) < 0, \tag{29}$$

$$\frac{\partial}{\partial \lambda'} \mathbb{E}[r^p(\alpha')] - r^p(\alpha^s) > 0. \tag{30}$$

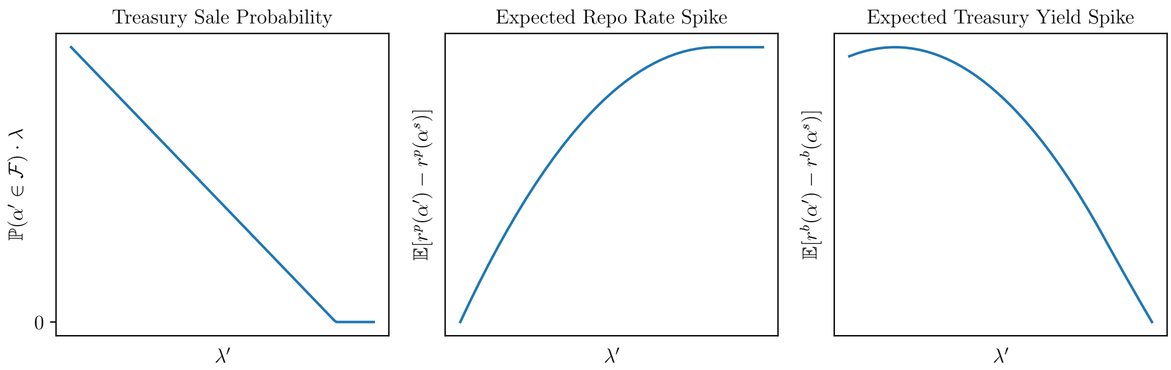

Figure 4: Simple Dynamic Model: Repo Supply Shock Duration. Shock duration is inverse to λ', the intensity of the Poisson process determining the return from a shock to the normal state. In particular, the expected duration of a shock is equal to $1/\lambda'$. Hence shock duration increases from the right to the left of the subplots. When shocks are short-lived, shadow banks are willing to take short-lived negative profits to avoid paying costly round-trip transaction costs. However, if shocks are expected to be very long lived, then a transaction becomes optimal to avoid negative spreads for long periods of time. Fire sales decrease repo demand within shocked states and hence the expected repo rate spike.

5 Balance Sheet Shocks in General Equilibrium

In this section, we present key insights from the model as a comparative statics exercise that abstracts from dynamics: state variables are constant over time ($\mathbf{x}_t = \mathbf{x} \ \ \forall t$).[12] Our primary focus is to understand how equilibrium prices and holdings evolve as a function of variables typically implicated in causing Treasury market disruptions, such as the size of foreign repo intermediation, the size of the TGA account, the size of the central bank balance sheet, and Treasury bond issuance. This approach enables an initial exploration of the effects of shocks, such as quarter ends, tax deadlines, quantitative tightening or easing, and fiscal expansion. In Section 4, we incorporate dynamic shocks back into the model and examine the impact of the expected severity and duration of these shocks on repo and Treasury markets in a fully dynamic setting. We begin by introducing a baseline without repo or reverse repo facilities and subsequently demonstrate how the introduction of these facilities alters the baseline equilibria.

[12]Implicitly, we assume that α_t is constant over time and the transaction cost κ is sufficiently high to keep the allocation of Treasury bonds constant over time ($w^b_t = w^b_{t\text{-}} = w^b$ and $\overline{w}^b_t = \overline{w}^b_{t\text{-}} = \overline{w}^b$).

5.1 Repo Market Imbalance

In this section, we first analyze the influence of repo demand and supply imbalances on equilibrium interest rates with and without standing repo and reverse repo facilities. We then investigate the impact of specific shocks on equilibrium prices and allocations.

No Standing Facilities We first characterize equilibrium interest rates in the absence of central bank facilities. That is, the central bank sets interest rates for its facilities that are never binding $r^{rp} = \infty$ and $r^{rrp} = -\infty$.

Lemma 2. *In an economy without facilities, liquidity services are maximized if and only if traditional banks are net lenders of repo and the IL constraint is not binding; that is,* $\ell(x^\star) > \ell(x)$ $\forall x, x^\star$ *if and only if* $\mathcal{M}(x^\star) \in \mathcal{U}$ *and* $\mathcal{M}(x) \notin \mathcal{U}$*.*

Proposition 2 shows that whenever traditional banks are net borrowers of repos or the IL constraint is binding, market forces cannot adjust in order to supply the first-best allocation of liquid assets to households. In that case, repo rates deviate from the interest on reserves in order to compensate households for having to provide more or less repo funding than their optimal portfolio composition. Lemma 3 demonstrates that high quantities of Treasury bonds outstanding $b_t - \underline{b}_t$, low future tax liabilities $\tau^h n^h$, low supply of reserves m, or high household preference for deposits α can lead to an upward deviation of repo rates from the interest on reserves, and vice versa.

Lemma 3. *In the absence of repo and reverse repo facilities:*

(i) $r^p > r^m$ *if and only if*

$$\underbrace{b - \underline{b} - w^b n - \overline{n}}_{\text{s-bank repo demand}} > \underbrace{(1-\alpha)(1+\tau^h)n^h + \kappa m}_{\text{highest repo supply at optimum}}; \tag{31}$$

(ii) $r^p < r^m$ *if and only if*

$$\underbrace{b - \underline{b} - w^b n - \overline{n}}_{\text{s-bank repo demand}} < \underbrace{(1-\alpha)(1+\tau^h)n^h}_{\text{lowest repo supply at optimum}}. \tag{32}$$

Lemma 3 tells us that when the demand for repo from shadow banks, $b - \underline{b} - w^b n - \overline{n}$, is high, traditional can provide the marginal funds, up to the IL constraint (κm). When the IL constraint is binding, repo rates must increase to incentivize households to provide more repo than their optimal portfolio allocation, which is given by $(1-\alpha)(1+\tau^h)n^h$. Conversely, when the supply of repo is too high, repo rates must drop to either incentivize households to hold less repo or compensate the balance sheet cost for traditional banks to fund themselves with repo.

Standing Central Bank Facilities We now characterize equilibria in settings with central bank repo and reverse repo facilities. For parsimony, we rule out the two extreme cases in which households would find it profitable to hold all their assets entirely at the reverse repo facility and in which traditional banks would be funded entirely at the repo facility. To provide policy-relevant insights on the design of facilities, we consider two types of standing repo facilities: one open to all agents (including shadow banks) and one open to banks only.

Lemma 4. *A reverse repo facility with rate r^{rrp} acts as a floor in the triparty repo market: $r^{pt} \geq r^{rrp}$. A repo facility with rate r^{rp} that is open only to traditional banks acts as a ceiling in the triparty repo market: $r^{pt} \leq r^{rp}$. A broad-access repo facility open to both traditional and shadow banks acts as a ceiling in the bilateral repo market: $r^{p} \leq r^{rp}$.*

Lemma 4 outlines specific outcomes for different central bank facility designs. First, a reverse repo facility offers a fixed rate of return option for households when investing in repos that is independent of the rate provided by traditional banks at the dealer subsidiary. Since the dealer's funding rate (triparty repo rate r^{pt}) is consistently lower than the dealer's lending rate (bilateral rate r^{p}) to compensate for balance sheet costs, r^{rrp} functions as a floor for the triparty repo market rate. Likewise, a repo facility accessible only to traditional banks serves as a ceiling on the dealer's funding rate (triparty rate), but not on the shadow banks' funding rate (bilateral rate). A bilateral rate higher than the repo facility rate would not incentivize traditional banks to borrow at the facility unless the triparty rate is also higher than the repo facility rate. However, if the repo facility is open to both traditional and shadow banks, shadow banks choose to borrow at the facility instead of traditional banks, as shadow banks are not subject to costly balance sheet constraints. Consequently, a broad access repo facility acts as a rate ceiling on the bilateral repo rate rather than the triparty repo rate.

5.2 Repo Shocks Decomposition

In this section, we present our comparative findings. We examine the behavior of repo rates and Treasury yields for various institutional settings under the following shocks: intermediation, tax deadline, fiscal expansion, and quantitative tightening. Our analysis allows a precise decomposition of the mechanisms leading a specific shock to a specific outcome and demonstrates that the effectiveness of facilities depends on the type of shock being considered. As previously mentioned, we assume that Treasury holdings remain fixed while varying other model parameters one by one. For the sake of simplicity in presenting this section's results, we set $w^b = 0$ without loss of generality.[13]

[13] We show in the online appendix that the results in this section are unaffected by setting w^b to another fixed value for comparative statics. As discussed in Section 5, with dynamic shocks, Treasury bond holdings w^b and $\overline{w}^b$ are determined endogenously as functions of the shock probability and the anticipated shock duration.

Intermediation Shock We begin by analyzing comparative static changes in the foreign dealer intermediation volumes f. This experiment is designed to capture the withdrawal from foreign dealer intermediaries at quarter-ends for window dressing their balance sheets, as studied by (Munyan, 2015) but could also represent any outright increase in repo intermediation cost, as could arise from changes in regulation for example or a negative shock to bank equity. Figure 5 presents the comparative statics outcomes for different levels of foreign dealer intermediation under various institutional frameworks. In red, we illustrate our baseline scenario without the RRP facility or IL constraint. As the foreign dealer sector contracts (i.e., moving leftward on the graphs), the triparty repo spread to IOR, $r^{pt} - r^m$, declines, allowing for an increase in intermediation spread $r^p - r^{pt}$ as compensation for a larger marginal balance sheet cost of banks $\chi\ell$ while maintaining the bilateral repo rate equal to IOR, $r^p = r^m$. This no-arbitrage condition is driven by Equation (22) in the absence of a binding IL constraint. Remarkably, in this baseline case, all other variables remain constant, including households' liquidity benefits, because the shock does not prompt any portfolio rebalancing from any agents.

The blue lines illustrate the situation when the Fed has a reverse repo facility in place, as has been the case since 2014 in the US. Following Lemma 4, the RRP facility establishes a lower bound on the triparty repo spread to IOR, $r^{pt} - r^m$. Upon reaching this limit, households exercise their option to lend repo directly to the Fed at the RRP rate, r^{rrp}, as shown in the Panel RRP Quantity. As seen in the third graph of Figure 5 , when the triparty repo rate reaches its RRP floor, banks begin lending in repo to shadow banks, thereby preventing bilateral repo rates from rising above IOR. This adjustment is made possible by the shift of households into RRP with the Fed, which reduces the reserve quantity on bank balance sheets and creates room for banks to lend. As noted by Diamond, Jiang, and Ma (2022), by occupying space on banks' balance sheets, reserves crowd out potential lending opportunities, so a decrease in reserves can benefit repo markets. In other words, our model clarifies that the "reserves-draining" repo lending from banks observed at quarter-end by Correa, Du, and Liao (2020) is a side-product of the concurrent surge in reverse repo facility volumes also documented in the same article. Examining the first diagram, which displays the spread between the bilateral repo rate and IOR, $r^p - r^m$, and traditional banks' repo lending positions, we observe that banks only carry out these operations until reaching the point f^{IL}. This point corresponds to the moment when the IL constraint becomes binding. Beyond that point, the reserve quantity on banks' balance sheets limits banks' ability to lend in repo, causing the bilateral repo rate to rise above IOR. This yield surge is the consequence of households having to adjust their portfolios to hold more repo, resulting in reduced liquidity benefits from their liquidity assets.

Following this reasoning, Proposition 3 proves that a repo spike requires the combination of three previously mentioned frictions: a binding IL, a positive balance sheet cost, and an RRP facility. In other words, relaxing any of those assumptions would make a repo spike impossible following a reduction in foreign intermediation.

Proposition 3. *Given*

$$b - m - a - w^b n - \overline{n} \leq (1-\alpha)(1+\tau^h)n^h + \kappa m, \tag{33}$$

the bilateral repo rate is above the interest on reserves, $r^p > r^m$, if and only if (i) IL is binding $\vartheta^m > 0$, (ii) balance sheet cost is positive $\chi > 0$, and (iii) RRP facility is binding $rrp > 0$.

Condition (33) is akin to Condition (31), but accounting for the presence of facilities, and guarantees that the baseline demand for repo is not already above the capacity of the system unconstrained by IL. The role of the RRP facility is particularly noteworthy; in its absence, the intermediation shock is absorbed by a continuous decrease in triparty rates, and the portfolio allocation of households remains close to the optimum. By establishing a lower bound on the triparty rates, the central bank introduces a market distortion by subsidizing triparty repo markets through direct provision of repo assets. This subsidy is a necessary condition for causing excessive household portfolio allocation to repo when the IL constraint becomes binding, as seen in Panel Liquidity Services. It is important to note that this misallocation does not necessarily result in a welfare loss, as the RRP facility also economizes on the balance sheet cost, which represents a deadweight loss in this economy.

We proceed to explore the implications of introducing a repo facility. Our analysis reveals that the facility's design is key to its efficacy. Notably, a repo facility that remains inaccessible to shadow banks, as by the current repo facility design at the Fed, is not effective in this scenario since a bank-intermediated repo facility fails to alleviate a spread increase caused by a congested dealer balance sheet (see Lemma 4). In contrast, a broad-access repo facility, when combined with a reverse repo facility, allows the central bank to effectively act as an intermediary in the repo markets, thereby preventing the repo rate from rising beyond the repo facility rate. This result echos the arguments from Duffie, Geithner, Parkinson, and Stein (2022) arguing in favor of broadening the access of the Fed's standing repo facility.

This impact of a repo facility accessible to banks can be observed with the yellow lines in Figure 5, where we verify that the bilateral repo rate does not surge above the RP facility rate. This cap is made feasible in this situation because the Fed effectively serves as an intermediary in the repo market by concurrently borrowing from households in triparty repo markets and lending to shadow banks in the bilateral repo, thereby economizing on dealers' balance sheet utilization.

Tax Deadline Shock We further investigate a scenario involving a repo supply shock, such as during tax deadlines when corporations utilize their cash balances in money market funds to meet their tax obligations. These tax payments are deposited into the TGA, as shown in Figure 6.

In contrast to the repo intermediation shock, a tax deadline shock does not lead to an increase in the repo intermediation spread (i.e., the difference between bilateral and triparty repo rates

$r^p - r^{pt}$) as dealers' balance sheets do not expand. Instead, the diminished repo supply exerts simultaneous upward pressure on both bilateral and triparty repo rates, causing them to move in tandem. Additionally, the inflow of reserves into the TGA reduces the supply of reserves accessible to banks, leading to tighter intraday regulatory restrictions, limiting traditional banks' capacity to lend in repo, and exacerbating the repo supply shock. This mechanism corresponds with the events of September 2019 and aligns with the findings of Correa, Du, and Liao (2020), which establish a connection between the TGA and repo rates. In this scenario, due to the net reduction in repo supply from households and the subsequent increase in triparty repo rates, the reverse repo facility does not come into play.

We also examine the introduction of a standing repo facility under this scenario with various access designs. Contrasting with the intermediation shock, we find that a repo facility, even if accessible only to banks, is sufficient to prevent repo rates from exceeding the facility rate (blue lines of Figure 6). Under this setting, traditional banks borrow repos from the central bank and lend them to shadow banks via dealer subsidiaries without an increase in the size of their balance sheet. The essential difference between the intermediation shock and the tax deadline shock lies in the fact that, for the latter, when banks borrow from the central bank to intermediate repos to shadow banks, they merely compensate for the diminished repo funding from households, so their balance sheets do not need to expand beyond the baseline case. With a broad access repo facility, shadow banks directly borrow from the central bank, thereby further economizing the traditional banks' balance sheets, making it a more efficient tool in our model.

Quantitative Tightening Shock We examine the impact of central bank balance sheet shocks on repo markets. Figure 7 illustrates that a reduction in the central bank balance sheet (moving leftward on the x-axes) simultaneously influences repo demand from shadow banks through Treasury supply and potential repo supply from traditional banks through reserves supply.[14]

Initially, as the central bank sells Treasuries, shadow banks increase their holdings of Treasuries and their demand for repo. As long as traditional banks hold enough reserves to satisfy their IL constraint, this increase in shadow banks' portfolios does not affect repo spreads ($r^p - r^m$), because the concurrent rundown of reserves frees up space on banks' balance sheets, enabling them to provide the necessary repo to shadow banks. Similar to previous shocks, issues arise once banks reach their IL constraint at point b^{IL} and cannot further lend in repo to shadow banks. Beyond this point, any additional reduction in the central bank's balance sheet results in an increase of both repo spreads $r^p - r^m$ and $r^{pt} - r^m$ to compensate households for shifting their portfolio away from the optimal portfolio composition and toward more repo lending (see Panel T-Banks Repo). In line with the tax deadline shock, we find that a reverse

[14] Although we interpret comparative statics as a reduction of the central bank balance sheet (QT), we stress that all insights have an inverse interpretation when the central bank increases its balance sheet size (QE), and acting as buyer-of-last-resort.

repo facility does not come into play as a result of a smaller central bank balance sheet size (blue lines of Figure 7). Instead, the RRP facility plays a significant role when the central bank balance sheet is large, transforming excess reserves into scarce repos that are in high demand from households.[15] Lastly, as with the tax deadline shock, a standing repo facility accessible only to banks is sufficient to prevent bilateral repo rates from spiking above the repo facility rate, but at a higher balance sheet cost compared to a broad-access facility (yellow lines of Figure 7).

Fiscal Expansion Shock In Figure 8, we investigate the influence of fiscal expansion shocks on repo and Treasuries markets. When not accompanied by additional purchases from the central bank, an increase in outstanding Treasuries shifts the balance towards higher repo demand, as Treasury holdings are primarily held by shadow banks and financed with repo. Consequently, this shock puts further upward pressure on repo rates and Treasury yields in a manner similar to the quantitative tightening shock discussed previously. Moreover, following the sale of issued Treasury securities, the TGA balance expands, displacing reserves and consequently tightening the IL constraint. As with the prior discussion, the reverse repo facility will come into play as a consequence of a reduction of Treasury supply rather than an increase.[16] Moreover, a repo facility available exclusively to traditional banks can help prevent repo spikes. However, similar to the central bank shock case, the most efficient approach involves opening the facility directly to shadow banks. This strategy economizes on bank balance sheet space and prevents repo rates from exceeding the repo facility rate.

6 Conclusion

This article proposes a dynamic model of the Treasury market that captures the various disruptions observed in recent years. It emphasizes the central bank's balance sheet, portfolio allocations, and regulatory frictions in shaping market stability. Our framework identifies the necessary frictions to explain disruptions, highlights the dual nature of reserve drawdowns, and investigates the effectiveness of repo facilities. To allow for a tractable exposition, our framework is nonetheless leaving out some elements that are likely to interact with the aforementioned results, including the absence of interest rate risk on long-term Treasuries, more realistic wealth dynamics, and additional regulatory pieces such as the liquidity coverage ratio. Exploring those interactions is left for future research. Overall, this study contributes to the policy debate and provides a foundation for future research on the sources of government securities market instability and the impact of regulation on government funding costs.

[15] This mechanism is investigated by d'Avernas and Vandeweyer (2023).

[16] Note that our focus is on longer-term Treasury securities that are financed through repo, as opposed to T-bills, which are directly held by money funds and serve as direct substitutes for repo. For an analysis of how the supply of T-bills impacts triparty repo rates, refer to the study conducted by d'Avernas and Vandeweyer (2023).

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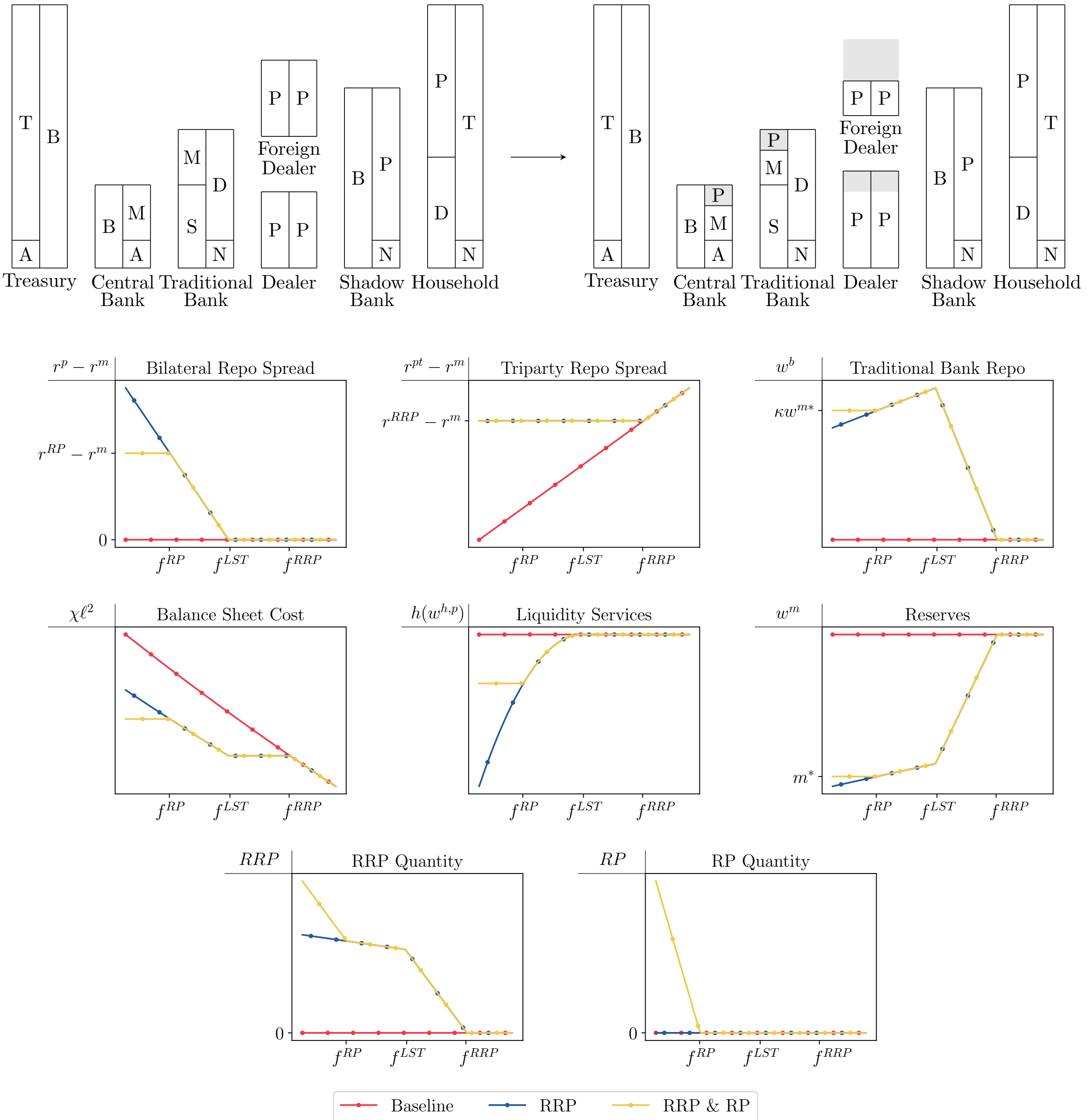

Figure 5: Intermediation Shock. The top panels show the impact of an intermediation shock, such as foreign dealer window-dressing on quarter-ends, on the balance sheets of the economy. T denotes the present value of future tax, B denotes Treasury bonds, N denotes net worth, M denotes reserves, K denotes capital, P denotes repo, and D denotes deposits. Comparative statics are displayed below the balance sheet diagrams. A negative shock to intermediation would be a move from the right side of a chart (high foreign dealer intermediation) to the left (low foreign dealer intermediation). f^{RRP} refers to the quantity of foreign repo intermediation at which $r^{pt} = r^{RRP}$ with no facilities, f^{IL} refers to the quantity of foreign repo intermediation at which the left and right-hand sides of Equation 31 are equalized with a central bank reverse repo facility in place, and f^{RP} refers to the quantity of foreign repo intermediation at which $r^p = r^{RP}$ with a reverse repo facility in place.

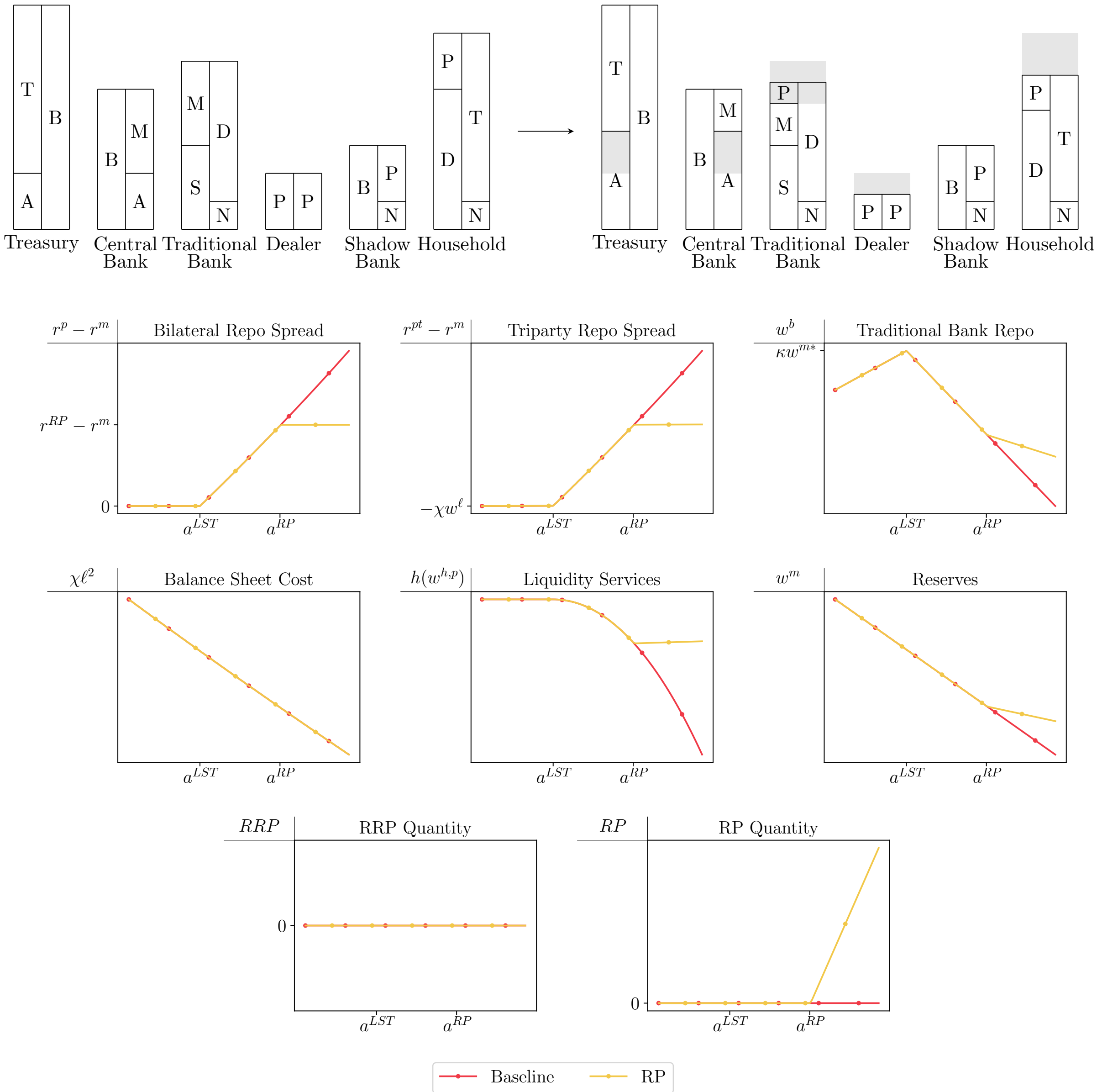

Figure 6: Tax Deadline Shock. Balance sheets prior to and after a tax deadline shock are shown on the top left and top right, respectively. T denotes the present value of future tax, B denotes Treasury bonds, N denotes net worth, M denotes reserves, K denotes capital, P denotes repo, and D denotes deposits. Comparative statics are displayed below the balance sheet diagrams. A positive shock to the Treasury General Account (TGA) would be a move from the left side of a chart to the right. a^{IL} refers to the size of the TGA at which the left and right hand sides of Equation 31 are equalized with a central bank reverse repo facility in place.

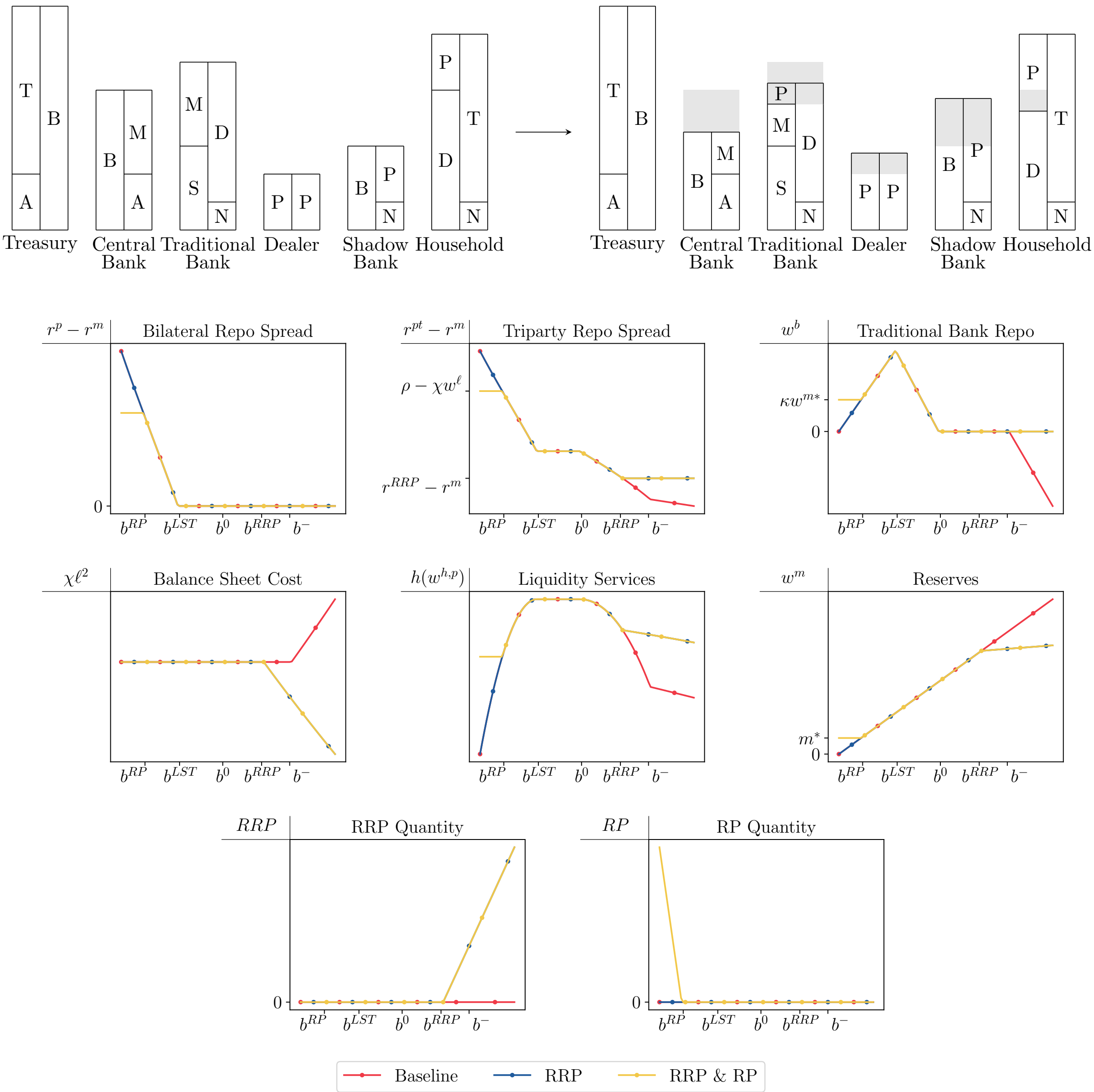

Figure 7: Central Bank Balance Sheet Shock. Balance sheets prior to and after a central bank balance sheet reduction are shown on the top left and top right, respectively. T denotes the present value of future tax, B denotes Treasury bonds, N denotes net worth, M denotes reserves, K denotes capital, P denotes repo, and D denotes deposits. Comparative statics are displayed below the balance sheet diagrams. A decrease in the central bank balance sheet would be a move from the right side of a chart to the left. b^{RRP} refers to the quantity of Treasury bonds held by the central bank at which $r^{pt} = r^{RRP}$ with no facilities, b^{IL} refers to the quantity of Treasury bonds held by the central bank at which the left and right hand sides of Equation 31 are equalized with a central bank reverse repo facility in place, and b^{RP} refers to the quantity of Treasury bonds held by the central bank at which $r^p = r^{rp}$ with a reverse repo facility in place.

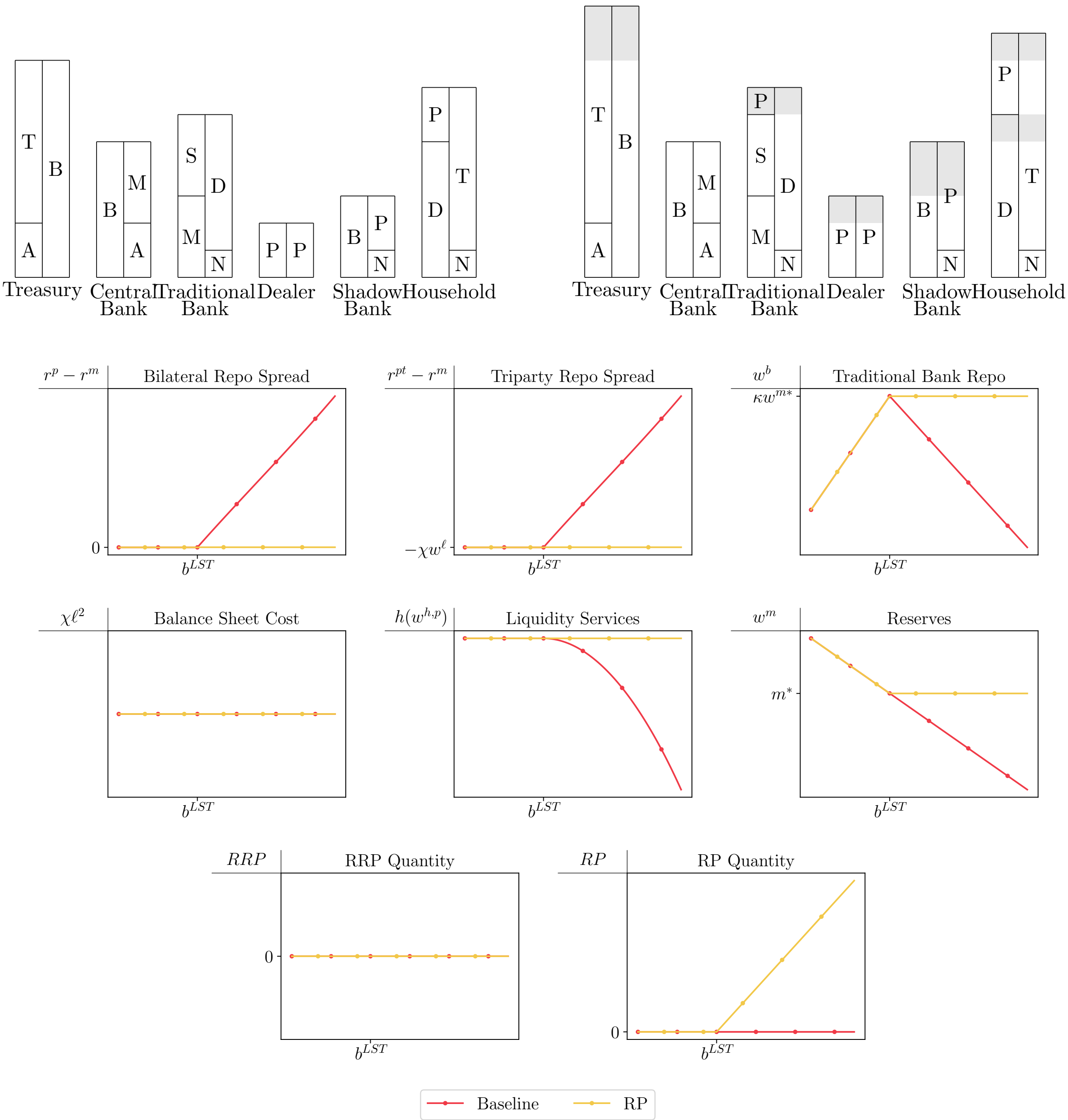

Figure 8: Fiscal Shock. Balance sheets prior to and after a fiscal shock, such as increased issuance of Treasuries, are shown on the top left and top right, respectively. T denotes the present value of future tax, B denotes Treasury bonds, N denotes net worth, M denotes reserves, K denotes capital, P denotes repo, and D denotes deposits. Comparative statics are displayed below the balance sheet diagrams. An increase in Treasuries issued would be a move from the left side of a chart to the right. b^* refers to the quantity of Treasury bonds issued by the Treasury at which the left and right hand sides of Equation 31 are equalized with a central bank reverse repo facility in place.

Appendices

A Proofs

Given our assumption on the law of motion of x_t, the functional form of the transaction cost, and aggregate wealth dynamics, equilibrium prices are only a function of $\mathbf{x}_t \equiv \{x_t, w^b_{t\text{-}}, \overline{w}^b_{t\text{-}}\}$. In the following proofs, we rewrite agents' problems in recursive form and drop the time subscript for ease of notation. We denote by $f(\cdot)$ the distribution of $\mathbf{x}'$ given the arrival of a Poisson shock from the steady state $\mathbf{x}^s$.

First, we guess and verify that the value functions have the following form:

$$V(n, w^b; \mathbf{x}) = \xi(\mathbf{x}) + \frac{\log(n)}{\rho} + \frac{\theta(\mathbf{x})w^b}{\rho}, \tag{34}$$

$$\overline{V}(\overline{n}, \overline{w}^b; \mathbf{x}) = \overline{\xi}(\mathbf{x}) + \frac{\log(\overline{n})}{\rho} + \frac{\overline{\theta}(\mathbf{x})\overline{w}^b}{\rho}, \tag{35}$$

$$V^h(n^h; \mathbf{x}) = \xi^h(\mathbf{x}) + (1+\beta)\frac{\log(n^h)}{\rho}. \tag{36}$$

Shadow Banks We can write the HJB for shadow banks as

$$\overline{V}(\overline{n}_-, \overline{w}^b_-; \mathbf{x}) = \max_{\overline{c}, \overline{w}^b, \overline{w}^p} \Big\{ \log(\overline{cn})dt + (1-\rho dt)(1-\lambda dt)\mathbb{E}_t\big[\overline{V}(\overline{n}+d\overline{n}, \overline{w}^b; \mathbf{x}+d\mathbf{x})|dN=0\big] + (1-\rho dt)\lambda(\mathbf{x})dt\mathbb{E}\big[\overline{V}(\overline{n}+d\overline{n}, \overline{w}^b; \mathbf{x}+d\mathbf{x})|dN=1\big]\Big\} \tag{37}$$

such that $\overline{w}^b = 1 + \overline{w}^p$ and

$$d\overline{n} = \Big(\overline{w}^b r^b - \overline{w}^p r^p - \overline{c}\Big)\overline{n}dt + (e^{-\nu|d\overline{w}^b|} - 1)\overline{n}. \tag{38}$$

Using Ito's lemma, the law of motion for $\mathbf{x}$, and the law of motion for $\overline{n}$, we can rewrite the HJB in equation (39) as

$$\begin{aligned}(\rho + \lambda(\mathbf{x}))&\overline{V}(\overline{n}, \overline{w}^b(\mathbf{x}); \mathbf{x}) \\ &= \log(\overline{c}(\mathbf{x})\overline{n}) + \overline{V}_n(\overline{n}, \overline{w}^b(\mathbf{x}); \mathbf{x})\Big(\overline{w}^b(\mathbf{x})r^b(\mathbf{x}) + (1-\overline{w}^b(\mathbf{x}))r^p(\mathbf{x}) - \overline{c}(\mathbf{x})\Big)\overline{n} \\ &+ \lambda(\mathbf{x})\int \overline{V}(\overline{n}e^{-\nu|\overline{w}^b(\mathbf{x}')-\overline{w}^b(\mathbf{x})|}, \overline{w}^b(\mathbf{x}'); \mathbf{x}')f(\mathbf{x}')d\mathbf{x}'.\end{aligned} \tag{39}$$

Substitute with the guess for $\overline{V}$ obtains

$$\begin{aligned}(\rho + \lambda(\mathbf{x}))&\overline{V}(\overline{n}, \overline{w}^b(\mathbf{x}); \mathbf{x}) \\ &= \log(\overline{c}(\mathbf{x})\overline{n}) + \frac{\overline{w}^b(\mathbf{x})r^b(\mathbf{x}) + (1-\overline{w}^b(\mathbf{x}))r^p(\mathbf{x}) - \overline{c}(\mathbf{x})}{\rho} \\ &+ \lambda(\mathbf{x})\int \left(\overline{\xi}(\mathbf{x}') + \frac{\log(\overline{n}e^{-\nu|\overline{w}^b(\mathbf{x}')-\overline{w}^b(\mathbf{x})|})}{\rho} + \frac{\overline{\theta}(\mathbf{x}')\overline{w}^b(\mathbf{x}')}{\rho}\right)f(\mathbf{x}')d\mathbf{x}'.\end{aligned} \tag{40}$$

As bankers can adjust their holdings of treasuries instantaneously by paying the transaction cost, the value function given $\overline{w}^b_-$ must be equal to the value that would obtain by changing the debt level to the optimum, that we denote $\overline{w}^{b\star}(\overline{w}^b_-;\mathbf{x})$:

$$\overline{V}(\overline{n}_-,\overline{w}^b_-;\mathbf{x}) = \max_{\overline{w}^b}\left\{\ \overline{V}(\overline{n}_-\overline{\iota}(\overline{w}^b_-,\overline{w}^b),\overline{w}^b;\mathbf{x})\right\} \tag{41}$$

$$= \overline{V}(\overline{n}_-\overline{\iota}(\overline{w}^b_-,\overline{w}^{b\star}(\overline{w}^b_-;\mathbf{x})),\overline{w}^{b\star}(\overline{w}^b_-;\mathbf{x});\mathbf{x}). \tag{42}$$

For ease of notation, we sometimes use the short notation $\overline{w}^{b\star} \equiv \overline{w}^{b\star}(\overline{w}^b_-;\mathbf{x})$ and $\iota^\star \equiv \iota(w^b_-, w^{b\star}(w^b_-;\mathbf{x}))$, where $\iota(w,w') \equiv e^{-\nu|w-w'|}$ is the adjustment cost function. Thus, $\overline{w}^{b\star}$ is determined by

$$\overline{V}_n(\overline{n}_-\iota(\overline{w}^b_-,\overline{w}^{b\star}),\overline{w}^{b\star};\mathbf{x})\iota_w(\overline{w}^b_-,\overline{w}^{b\star})\overline{n}_- + \overline{V}_w(\overline{n}_-\iota(\overline{w}^b_-,\overline{w}^{b\star}),\overline{w}^{b\star};\mathbf{x}) = 0, \tag{43}$$

where we use the notation $f_x = \partial f/\partial x$ for partial derivatives. Substituting for the guess for $\overline{V}$, we get

$$-\nu\,\mathrm{sign}(\overline{w}^{b\star} - \overline{w}^b_-) + \overline{\theta}(\mathbf{x}) = 0. \tag{44}$$

Then, we can write the optimal weight on treasuries as follows:

$$\overline{w}^{b\star}(\overline{w}^b;\mathbf{x}) = \begin{cases} 1 & \text{if } \ \overline{\theta}(\mathbf{x}) < -\nu \\ [1,\overline{w}^b] & \text{if } \ \overline{\theta}(\mathbf{x}) = -\nu, \\ \overline{w}^b & \text{if } \ -\nu < \overline{\theta}(\mathbf{x}) < \nu, \\ [\overline{w}^b,\infty] & \text{if } \ \overline{\theta}(\mathbf{x}) = \nu. \end{cases} \tag{45}$$

We omit the case for $\overline{\theta}(\mathbf{x}) > \nu$ as this would lead to an infinite holding of treasuries, which is infeasible in equilibrium.

The first-order condition yields $\overline{c}(\mathbf{x}) = \rho$. If $|\overline{w}^b(\mathbf{x}') - \overline{w}^b(\mathbf{x})| > 0$, using the envelope condition with respect to w^b, we get

$$(\rho+\lambda(\mathbf{x}))\overline{\theta}(\mathbf{x}) = r^b(\mathbf{x}) - r^p(\mathbf{x}) \tag{46}$$

$$+ \lambda(\mathbf{x})\frac{\partial}{\partial \overline{w}^b(\mathbf{x})}\int\left(\log(\overline{n}e^{-\nu|\overline{w}^b(\mathbf{x}')-\overline{w}^b(\mathbf{x})|}) + \overline{\theta}(\mathbf{x}')\overline{w}^b(\mathbf{x}')\right)f(\mathbf{x}')d\mathbf{x}'. \tag{47}$$

For $\mathbf{x} = \mathbf{x}^s$, this becomes

$$(\rho+\lambda)\overline{\theta}(\mathbf{x}^s) = r^b(\mathbf{x}^s) - r^p(\mathbf{x}^s) \tag{48}$$

$$+ \lambda\nu\int_{\overline{w}^b(\mathbf{x}')\neq\overline{w}^b(\mathbf{x}^s)} \mathrm{sign}(\overline{w}^b(\mathbf{x}') - \overline{w}^b(\mathbf{x}^s))f(\mathbf{x}')d\mathbf{x}' \tag{49}$$

$$+ \lambda\int_{\overline{w}^b(\mathbf{x}')=\overline{w}^b(\mathbf{x}^s)} \overline{\theta}(\mathbf{x}')f(\mathbf{x}')d\mathbf{x}'. \tag{50}$$

For $\mathbf{x} \neq \mathbf{x}^s$, since after a Poisson shock α is guaranteed to be $\mathbf{x}^s$, we get

$$(\rho+\lambda')\overline{\theta}(\mathbf{x}) = r^b(\mathbf{x}) - r^p(\mathbf{x}) \tag{51}$$

$$+ \lambda'\nu\,\mathrm{sign}(\overline{w}^b(\mathbf{x}^s) - \overline{w}^b(\mathbf{x}))\mathbb{1}\{\overline{w}^b(\mathbf{x}^s) \neq \overline{w}^b(\mathbf{x})\} \tag{52}$$

$$+ \lambda'\overline{\theta}^b(\mathbf{x}^s)\mathbb{1}\{\overline{w}^b(\mathbf{x}^s) = \overline{w}^b(\mathbf{x})\}. \tag{53}$$

Traditional Banks Similarly, we can write the HJB for traditional banks as

$$\begin{aligned}
V(n_-,&w^b_-(\mathbf{x});\mathbf{x}) \\
&= \max_{c,w^s,w^b,w^m,w^d,w^p} \Big\{ \log(cn)dt \\
&\quad + (1-\rho dt)(1-\lambda(\mathbf{x})dt)\mathbb{E}_t\big[V(n+dn,w^b;\mathbf{x}+d\mathbf{x})|dN=0\big] \\
&\quad + (1-\rho dt)\lambda(\mathbf{x})dt\mathbb{E}\big[V(n+dn,w^b;\mathbf{x}+d\mathbf{x})|dN=1\big]\Big\} \qquad (54)
\end{aligned}$$

such that $w^k + w^b + w^m + w^p = 1 + w^d$,

$$\begin{aligned}
dn = \Big(&w^k r^k(\mathbf{x}) + w^b r^b(\mathbf{x}) + w^m r^m(\mathbf{x}) + w^p r^p(\mathbf{x}) \\
&-w^d r^d(\mathbf{x}) + w^x(r^p(\mathbf{x}) - r^{pt}(\mathbf{x})) - c\Big) ndt \qquad (55)\\
&- \frac{\chi}{2}\ell^2 ndt + (e^{-\nu|dw^b|} - 1)n, \qquad (56)
\end{aligned}$$

and

$$w^p \leq \kappa w^m, \qquad (57)$$

where

$$\ell \equiv w^d + w^x - \min\{0, w^p\}. \qquad (58)$$

As before, the value function given w^b_- must be equal to the value that would obtain by changing the Treasury bond holdings to the optimum, that we denote $w^{b\star}(w^b_-;\mathbf{x})$. For ease of notation, we sometimes use the short notation $w^{b\star} \equiv w^{b\star}(w^b_-;\mathbf{x})$, and $\iota^\star \equiv \iota(w^b_-, w^{b\star}(w^b_-;\mathbf{x}))$. That is,

$$\begin{aligned}
V(n_-, w^b_-;\mathbf{x}) &= \max_{w^b \geq 0} \Big\{ V(n_-\iota(w^b), w^b;\mathbf{x})\Big\} \qquad (59)\\
&= V(n_-\iota^\star, w^{b\star};\mathbf{x}). \qquad (60)
\end{aligned}$$

Thus, $w^{b\star}$ is determined by

$$V_n(n_-\iota^\star, w^{b\star};\mathbf{x})\iota^\star_{w^b} n_- + V_{w^b}(n_-\iota^\star, w^{b\star};\mathbf{x}) = 0. \qquad (61)$$

Substituting for the guess for V, we get

$$-\nu\,\mathrm{sign}(w^{b\star} - w^b_-) + \theta(\mathbf{x}) = 0 \qquad (62)$$

Thus,

$$w^{b\star}(w^b;\mathbf{x}) = \begin{cases} 0 & \text{if} \quad \theta(\mathbf{x}) < -\nu \\ [0, w^b] & \text{if} \quad \theta(\mathbf{x}) = -\nu, \\ w^b & \text{if} \quad -\nu < \theta(\mathbf{x}) < \nu, \\ [w^b, \infty] & \text{if} \quad \theta(\mathbf{x}) = \nu. \end{cases} \qquad (63)$$

Using the same steps as for the shadow banks and substitute with the guess for V obtains

$$(\rho+\lambda(\mathbf{x}))V(n,w^b(\mathbf{x});\mathbf{x}) = \log(c(\mathbf{x})n) + \frac{\mu^n(\mathbf{x})}{\rho} + \lambda(\mathbf{x})\int\left(\xi(\mathbf{x}') + \frac{\log(ne^{-\nu|w^b(\mathbf{x}')-w^b(\mathbf{x})|})}{\rho} + \frac{\theta(\mathbf{x}')w^b(\mathbf{x}')}{\rho}\right)f(\mathbf{x}')d\mathbf{x}' + \vartheta^m(\mathbf{x})(\kappa w^m(\mathbf{x}) - w^p(\mathbf{x})), \tag{64}$$

where

$$\mu^n(\mathbf{x}) \equiv w^k(\mathbf{x})r^k(\mathbf{x}) + w^b(\mathbf{x})r^b(\mathbf{x}) + w^m(\mathbf{x})r^m(\mathbf{x}) + w^p(\mathbf{x})r^p(\mathbf{x}) - w^d(\mathbf{x})r^d(\mathbf{x}) \tag{65}$$

$$+ w^x(r^p(\mathbf{x}) - r^{pt}(\mathbf{x})) - c(\mathbf{x}) - \frac{\chi}{2}\ell(\mathbf{x})^2 \tag{66}$$

and where $\vartheta^m(\mathbf{x})$ is the Lagrange multiplier on the constraint $\kappa w^m(\mathbf{x}) \geq w^p(\mathbf{x})$. Thus, given that in equilibrium $w^d > 0$, the first order condition for c, w^k, w^m and w^p are given by

$$c(\mathbf{x}) = \rho, \tag{67}$$

$$r^k(\mathbf{x}) - r^d(\mathbf{x}) = \chi\ell(\mathbf{x}), \tag{68}$$

$$r^m(\mathbf{x}) - r^d(\mathbf{x}) = \chi\ell(\mathbf{x}) - \kappa\vartheta^m(\mathbf{x}), \tag{69}$$

$$r^p(\mathbf{x}) - r^{pt}(\mathbf{x}) = \chi\ell(\mathbf{x}), \tag{70}$$

$$r^p(\mathbf{x}) - r^d(\mathbf{x}) = \begin{cases} \chi\ell(\mathbf{x}) + \vartheta^m(\mathbf{x}) & \text{if } w^p(\mathbf{x}) > 0 \\ 0 & \text{if } w^p(\mathbf{x}) < 0 \end{cases} \tag{71}$$

Thus,

$$w^p(\mathbf{x}) = \begin{cases} \in (-\infty, 0] & \text{if } r^p(\mathbf{x}) - r^d(\mathbf{x}) = 0 \\ 0 & \text{if } 0 < r^p(\mathbf{x}) - r^d(\mathbf{x}) < \chi\ell(\mathbf{x}) \\ \in [0, \kappa w^m(\mathbf{x})] & \text{if } r^p(\mathbf{x}) - r^d(\mathbf{x}) = \chi\ell(\mathbf{x}) \\ = \kappa w^m(\mathbf{x}) & \text{if } r^p(\mathbf{x}) - r^d(\mathbf{x}) > \chi\ell(\mathbf{x}). \end{cases} \tag{72}$$

As before, the envelope condition yields

$$(\rho+\lambda(\mathbf{x}))\theta(\mathbf{x}) = r^b(\mathbf{x}) - r^d(\mathbf{x}) - \chi\ell(\mathbf{x}) \tag{73}$$

$$+ \lambda(\mathbf{x})\frac{\partial}{\partial w^b(\mathbf{x})}\int\left(\log(ne^{-\nu|w^b(\mathbf{x}')-w^b(\mathbf{x})|}) + \theta(\mathbf{x}')w^b(\mathbf{x}')\right)f(\mathbf{x}')d\mathbf{x}'. \tag{74}$$

For $\mathbf{x} = \mathbf{x}^s$, this becomes

$$(\rho+\lambda)\theta(\mathbf{x}^s) = r^b(\mathbf{x}^s) - r^d(\mathbf{x}^s) - \chi\ell(\mathbf{x}^s) \tag{75}$$

$$+ \lambda\nu\int_{w^b(\mathbf{x}')\neq w^b(\mathbf{x}^s)} \text{sign}(w^b(\mathbf{x}') - w^b(\mathbf{x}^s))f(\mathbf{x}')d\mathbf{x}' \tag{76}$$

$$+ \lambda\int_{w^b(\mathbf{x}')=w^b(\mathbf{x}^s)} \theta(\mathbf{x}')f(\mathbf{x}')d\mathbf{x}'. \tag{77}$$

For $\mathbf{x} \neq \mathbf{x}^s$, since after a Poisson shock α is guaranteed to be $\mathbf{x}^s$, we get

$$(\rho + \lambda')\theta(\mathbf{x}) = r^b(\mathbf{x}) - r^d(\mathbf{x}) - \chi\ell(\mathbf{x}) \tag{78}$$

$$+ \lambda'\nu \operatorname{sign}(w^b(\mathbf{x}^s) - w^b(\mathbf{x}))\mathbb{1}\{w^b(\mathbf{x}^s) \neq w^b(\mathbf{x})\} \tag{79}$$

$$+ \lambda'\theta(\mathbf{x}^s)\mathbb{1}\{w^b(\mathbf{x}^s) = w^b(\mathbf{x})\}. \tag{80}$$

Households Similarly, we can write the HJB for households as

$$V^h(n^h;\mathbf{x}) = \max_{c^h, w^{h,i}, w^{h,d}, w^{h,p}} \Big\{ \log(c^h n^h)dt + \beta \log(h(w^{h,p}, w^{h,d}; \alpha)n^h)dt + (1-\rho dt)(1-\lambda(\mathbf{x})dt)\mathbb{E}_t\big[V^h(n^h + dn^h; \mathbf{x} + d\mathbf{x})|dN = 0\big] + (1-\rho dt)\lambda(\mathbf{x})dt\mathbb{E}\big[V^h(n^h + dn^h; \mathbf{x} + d\mathbf{x})|dN = 1\big]\Big\} \tag{81}$$

where

$$h(w^{h,p}, w^{h,d}; \alpha) = (w^{h,d})^\alpha (w^{h,p})^{1-\alpha} \tag{82}$$

and such that $w^{h,p} + w^{h,d} = 1 + \tau^h$ and

$$dn^h = (w^{h,d}r^d(\mathbf{x}) + w^{h,p}r^{pt}(\mathbf{x}) - c^h)n^h dt. \tag{83}$$

We can rewrite the HJB as follows:

$$(\rho + \lambda(\mathbf{x}))V^h(n^h;\mathbf{x}) = \max_{c^h, w^{h,p}, w^{h,d}} \Big\{ \log(c^h n^h) + \beta \log(h(w_u^{h,p}, w_u^{h,d}; \alpha)n^h) + (1+\beta)\frac{\mu^{h,n}(\mathbf{x})}{\rho} + \lambda(\mathbf{x}) \int \left(\xi^h(\mathbf{x}') + \frac{(1+\beta)\log(n^h)}{\rho}\right) f(\mathbf{x}')d\mathbf{x}' \Big\}, \tag{84}$$

where

$$\mu^{h,n} = w^{h,d}r^d(\mathbf{x}) + w^{h,p}r^{pt}(\mathbf{x}) - c. \tag{85}$$

The first-order conditions for households are given by

$$c^h(\mathbf{x}) = \frac{\rho}{1+\beta}, \tag{86}$$

$$r^{pt}(\mathbf{x}) - r^d(\mathbf{x}) = \rho\frac{\beta}{1+\beta}\left(\frac{\alpha}{w^{h,d}(\mathbf{x})} - \frac{1-\alpha}{w^{h,p}(\mathbf{x})}\right). \tag{87}$$

B Proofs of section 4

In the following, we write variables as functions of the state variable α_t instead of $\mathbf{x}_t$. We begin by a set of intermediary results before providing the proofs of our propositions.

Lemma A1. *If there exists α' such that $0 < w^b(\alpha^s) < w^b(\alpha')$, then $\theta(\alpha^s) \leq -\overline{\theta}(\alpha^s) = -\nu$, $\theta(\alpha') = -\overline{\theta}(\alpha') = \nu$, $\overline{w}^b(\alpha^s) > \overline{w}^b(\alpha')$, and there does not exist a α' such that $w^b(\alpha') < w^b(\alpha^s)$ or $\overline{w}^b(\alpha^s) < \overline{w}^b(\alpha')$.*

Proof. From the envelope condition for w^b in equation (63), we have that for the traditional bank to be incentivized to increase its holding of treasuries and pay the adjustment cost when moving from α^s to α' and vice versa, then it must be that $\theta(\alpha') = \nu$ and $\theta(\alpha^s) \leq -\nu$. Thus, there cannot exist another state such that $w^b(\alpha') < w^b(\alpha^s)$.

The market clearing condition for the treasury market is given by

$$w^b(\alpha)n + \overline{w}^b(\alpha)\overline{n} + \underline{b} = b \quad \forall \alpha. \tag{88}$$

Thus, the reverse must be true for $\overline{w}^b(\alpha)$ and $\overline{\theta}(\alpha)$. □

Steady State: $\alpha = \alpha^s$. Given Lemma A1, the restriction that $w^b(\alpha^s) > 0$, and the envelope conditions for traditional and shadow banks lead to the following expressions for rates relative to $r^d(\alpha^s)$:

$$r^b(\alpha^s) - r^d(\alpha^s) = \chi\ell(\alpha^s) - (\rho+\lambda)\nu - \lambda\nu\mathbb{P}(\alpha \in \mathcal{F}) - \lambda\int_{\alpha' \notin \mathcal{F}} \theta(\alpha')f(\alpha')d\alpha', \tag{89}$$

$$r^b(\alpha^s) - r^p(\alpha^s) = (\rho+\lambda)\nu + \lambda\nu\mathbb{P}(\alpha \in \mathcal{F}) - \lambda\int_{\alpha' \notin \mathcal{F}} \overline{\theta}(\alpha')f(\alpha')d\alpha'. \tag{90}$$

Thus,

$$r^p(\alpha^s) - r^d(\alpha^s) = \chi\ell(\alpha^s) - 2\nu(\rho+\lambda) - 2\lambda\nu\mathbb{P}(\alpha \in \mathcal{F}) + \lambda\int_{\alpha' \notin \mathcal{F}} (\overline{\theta}(\alpha') - \theta(\alpha'))f(\alpha')d\alpha'. \tag{91}$$

Thus, $0 \leq r^p(\alpha^s) - r^d(\alpha^s) < \chi\ell(\alpha^s)$ as $\overline{\theta}(\alpha') - \theta(\alpha') \leq 2\nu$. This further implies that $w^p(\alpha^s) \leq 0$ from the first-order-condition for w^p. Given our restriction that $w^p(\alpha^s) \geq 0$, we get $w^p(\alpha^s) = 0$. From the first-order condition for $w^x(\alpha^s)$, we get

$$r^{pt}(\alpha^s) - r^d(\alpha^s) = -2\nu(\rho+\lambda) - 2\lambda\nu\mathbb{P}(\alpha \in \mathcal{F}) + \lambda\int_{\alpha' \notin \mathcal{F}} (\overline{\theta}(\alpha') - \theta(\alpha'))f(\alpha')d\alpha'. \tag{92}$$

For ease of notation, define

$$\Theta^s \equiv -2\nu(\rho+\lambda) - 2\lambda\nu\mathbb{P}(\alpha \in \mathcal{F}) + \lambda\int_{\alpha' \notin \mathcal{F}} (\overline{\theta}(\alpha') - \theta(\alpha'))f(\alpha')d\alpha'. \tag{93}$$

Given $w^p(\alpha^s) = 0$, from the repo market clearing condition, we obtain $w^{h,p}(\alpha^s)n^h = \overline{w}^p(\alpha^s)\overline{n}$ and $w^{h,p}$ is determined by $r^{pt}(\alpha^s) - r^d(\alpha^s) = \Theta^s$. For ease of notation, we define $\mathcal{W}$ as

$$\mathcal{W}(s,\alpha) \equiv \frac{s(1+\beta)(1+\tau^h) - \rho\beta + \mathcal{G}(s,\alpha)}{2s(1+\beta)} \tag{94}$$

where

$$\mathcal{G}(s,\alpha) \equiv \sqrt{(\rho\beta)^2 + s^2(1+\beta)^2(1+\tau^h)^2 + 2\rho\beta s(1+\beta)(1+\tau^h)(1-2\alpha)}. \tag{95}$$

Then $\mathcal{W}(r^{pt} - r^d, \alpha^s)$ is the solution[17] of equation (87) for $w^{h,p}$ in terms of a spread $r^pt - r^d$.

[17]Only this root is relevant as the other root implies that either $w^{h,p}$ or $w^{h,d}$ is negative.

a proof of why the other root is invalid. The other root is

$$\mathcal{W}' \equiv \frac{s(1+\beta)(1+\tau^h) - \rho\beta - \mathcal{G}(s,\alpha)}{2s(1+\beta)} \tag{96}$$

If $s > 0$, then we have

$$\mathcal{W}(s,\alpha) + \mathcal{W}' = -\frac{\rho\beta(1-\alpha)(1+\tau^h)}{s(1+\beta)} < 0 \tag{97}$$

Hence the smaller root $\mathcal{W}' < 0$.

If $s < 0$, then we have

$$\mathcal{G}(s,\alpha) > \sqrt{(\rho\beta)^2 + s^2(1+\beta)^2(1+\tau^h)^2 + 2\rho\beta s(1+\beta)(1+\tau^h)} = |s(1+\beta)(1+\tau^h) + \rho\beta|$$

given that $\alpha \in (0,1)$, which further implies that

$$\begin{aligned}
\mathcal{W}' &= \frac{s(1+\beta)(1+\tau^h) - \rho\beta - \mathcal{G}(s,\alpha)}{2s(1+\beta)} \\
&> \frac{s(1+\beta)(1+\tau^h) - \rho\beta - |s(1+\beta)(1+\tau^h) + \rho\beta|}{2s(1+\beta)} \\
&= \frac{\min\{s(1+\beta)(1+\tau^h), -\rho\beta\}}{s(1+\beta)} = \max\left\{(1+\tau^h), -\frac{\rho\beta}{s(1+\beta)}\right\} \geq (1+\tau^h)
\end{aligned}$$

Hence, $1 + \tau^h - \mathcal{W}' < 0$. The root is also invalid.

Then, we have

$$w^{h,p}(\alpha^s) = \mathcal{W}(\Theta^s, \alpha^s) \tag{98}$$

which, combined with the repo market clearing condition, yields

$$\overline{w}^p(\alpha^s)\overline{n} = \mathcal{W}(\Theta^s, \alpha^s)n^h. \tag{99}$$

Then, combined with the the shadow bank balance sheet constraint and Treasury bond market clearing condition, we get

$$\overline{w}^b(\alpha^s)\overline{n} = \overline{n} + \mathcal{W}(\Theta^s, \alpha^s)n^h \tag{100}$$

and

$$w^b(\alpha^s)n = b - \underline{b} - \overline{n} - \mathcal{W}(\Theta^s, \alpha^s)n^h. \tag{101}$$

States with the IL Constraint Not Binding: $\alpha'|\vartheta^m(\alpha') = 0$. Consider some shock $\alpha' \in (\alpha^s, 1)$ such that the IL constraint does not bind. First, let us assume that there are no fire sale: $w^b(\alpha') = w^b(\alpha^s)$ and $\overline{w}^b(\alpha') = \overline{w}^b(\alpha^s)$. From the envelope conditions of traditional and shadow banks, we get

$$(\rho + \lambda')\theta(\alpha') = r^b(\alpha') - r^d(\alpha') - \chi\ell(\alpha') - \lambda'\nu, \tag{102}$$

$$(\rho + \lambda')\overline{\theta}(\alpha') = r^b(\alpha') - r^p(\alpha') + \lambda'\nu, \tag{103}$$

and

$$(\rho + \lambda')(\theta(\alpha') - \overline{\theta}(\alpha')) = r^p(\alpha') - r^d(\alpha') - \chi\ell(\alpha') - 2\lambda'\nu. \tag{104}$$

From the first-order condition for $w^x(\alpha')$, we get

$$(\rho + \lambda')(\theta(\alpha') - \overline{\theta}(\alpha')) = r^{pt}(\alpha') - r^d(\alpha') - 2\lambda'\nu. \tag{105}$$

Assume that $(\rho + \lambda')(\theta(\alpha') - \overline{\theta}(\alpha')) + 2\lambda'\nu > 0$ such that $r^{pt}(\alpha') > r^d(\alpha')$. Then, from the traditional bank first-order conditions, we get $\vartheta^m(\alpha') > 0$. Thus, it must be that $(\rho+\lambda')(\theta(\alpha')-\overline{\theta}(\alpha')) + 2\lambda'\nu \leq 0$.

Define $\Theta' \equiv (\rho + \lambda')(\theta(\alpha') - \overline{\theta}(\alpha')) + 2\lambda'\nu$. Since $\overline{w}^b(\alpha') = \overline{w}^b(\alpha^s)$, we get

$$\overline{w}^p(\alpha')\overline{n} = \mathcal{W}(\Theta^s, \alpha^s)n^h, \tag{106}$$

$$w^{h,p}(\alpha') = \mathcal{W}(\Theta', \alpha'). \tag{107}$$

Combining the bilateral and triparty repo market clearing conditions then yields

$$w^p(\alpha')n = \left(\mathcal{W}(\Theta^s, \alpha^s) - \mathcal{W}(\Theta', \alpha')\right) n^h. \tag{108}$$

We have two cases.

- Case $\Theta' < 0$. Then $r^{pt}(\alpha') < r^d(\alpha')$ and $w^p(\alpha') = 0$ by traditional bank first-order condition. Then $\mathcal{W}(\Theta^s, \alpha^s) = \mathcal{W}(\Theta', \alpha')$, which pins down the value of Θ'. Thus, $\mathcal{M}(\alpha') \in \mathcal{S}$.
- Case $\Theta' = 0$. Thus, $r^{pt}(\alpha') - r^d(\alpha') = 0$ and $w^p(\alpha') \in [0, \kappa w^m(\alpha')]$. Thus, $\mathcal{M}(\alpha') \in \mathcal{U}$.

In the first case, the spread between repo and deposit rates change from Θ^s to Θ' to compensate households for holding the same level of repo as in the normal state despite the preference shock. In the second case we have traditional banks acting as the marginal lenders of repo and household reach their optimal portfolio allocation.

We now verify the assumption that there is no sale of Treasury bonds with the IL constraint nonbinding. Assume by way of contradiction that there is a sale and that $\vartheta^m = 0$. Then by the envelope conditions we have

$$r^{pt}(\alpha') - r^d(\alpha') = 2\nu(\rho + \lambda') + 2\lambda'\nu. \tag{109}$$

From the traditional bank first-order conditions we have that $r^{pt}(\alpha') - r^d(\alpha') = 0$ if $w^p(\alpha') > 0$ and $r^{pt}(\alpha') - r^d(\alpha') = -\chi\ell(\alpha')$ if $w^p(\alpha') < 0$. Thus, $r^{pt}(\alpha') - r^d(\alpha') \leq 0$ necessarily. Then we have a contradiction as $2\nu(\rho + \lambda') + 2\lambda'\nu > 0$.

States with the IL Constraint Binding: $\alpha''|\vartheta^m(\alpha'') > 0$. Consider some shock $\alpha'' \in (\alpha^s, 1)$ such that the IL constraint binds. Then we have $w^p(\alpha'') = \kappa w^m(\alpha'')$ and $\vartheta^m > 0$. Therefore, from the first-order conditions of the traditional banks, it must be that $r^p(\alpha'') - r^d(\alpha'') > \chi\ell(\alpha'')$ and $r^{pt}(\alpha'') - r^d(\alpha'') > 0$. Assuming that no fire sales occurs, the envelope conditions yield

$$r^b(\alpha'') - r^d(\alpha'') = (\rho + \lambda')\theta(\alpha'') + \chi\ell(\alpha'') + \lambda'\nu, \tag{110}$$

$$r^b(\alpha'') - r^p(\alpha'') = (\rho + \lambda')\overline{\theta}(\alpha'') - \lambda'\nu. \tag{111}$$

Thus,

$$r^p(\alpha'') - r^d(\alpha'') = (\rho + \lambda')(\theta(\alpha'') - \overline{\theta}(\alpha'')) + \chi\ell(\alpha'') + 2\lambda'\nu. \tag{112}$$

From the traditional bank first-order condition for $w^x(\alpha'')$ we get

$$r^{pt}(\alpha'') - r^d(\alpha'') = (\rho + \lambda')(\theta(\alpha'') - \overline{\theta}(\alpha'')) + 2\lambda'\nu. \tag{113}$$

For ease of notation, define

$$\Theta'' \equiv (\rho + \lambda')(\theta(\alpha'') - \overline{\theta}(\alpha'')) + 2\lambda'\nu. \tag{114}$$

Given that $\vartheta^m(\alpha'') > 0$, $r^{pt}(\alpha'') - r^d(\alpha'') > 0$ and $\Theta'' > 0$. Since $\overline{w}^b(\alpha'') = \overline{w}^b(\alpha^s)$, then

$$\overline{w}^b(\alpha'')\overline{n} = \overline{n} + \mathcal{W}(\Theta^s, \alpha^s)n^h, \tag{115}$$

$$\overline{w}^p(\alpha'')\overline{n} = \mathcal{W}(\Theta^s, \alpha^s)n^h. \tag{116}$$

From the household first-order conditions, we get

$$w^{h,p}(\alpha'') = \mathcal{W}(\Theta'', \alpha''), \tag{117}$$

which, combined with the bilateral and triparty repo market clearing conditions, yields

$$w^p(\alpha'')n = \big(\mathcal{W}(\Theta^s, \alpha^s) - \mathcal{W}(\Theta'', \alpha'')\big)\, n^h. \tag{118}$$

Since $w^p(\alpha'') = \kappa m/n$, we can solve for Θ''.

If the solution gives $(\Theta'' - 2\lambda'\nu)/(\rho + \lambda') = \theta^b(\alpha'') - \overline{\theta}^b(\alpha'') > 2\nu$, then a fire sale occurs. Guess that $\overline{w}^b(\alpha'') > 1$. In this case, we get

$$r^{pt}(\alpha'') - r^d(\alpha'') = 2\nu(\rho + \lambda') + 2\lambda'\nu \tag{119}$$

and

$$r^p(\alpha'') - r^d(\alpha'') = 2\nu(\rho + \lambda') + 2\lambda'\nu + \chi\ell(\alpha''). \tag{120}$$

From the household first-order conditions, we get

$$w^{h,p}(\alpha'') = \mathcal{W}(2\nu(\rho + \lambda') + 2\lambda'\nu, \alpha''), \tag{121}$$

Since the IL constraint is binding, we get

$$w^p(\alpha'') = \kappa m/n, \tag{122}$$

which, combined with the bilateral and triparty repo market clearing conditions, yields

$$\overline{w}^p(\alpha'')\overline{n} = \mathcal{W}(2\nu(\rho + \lambda') + 2\lambda'\nu, \alpha'')n^h + \kappa m \tag{123}$$

and

$$\overline{w}^b(\alpha'')\overline{n} = \overline{n} + \mathcal{W}(2\nu(\rho + \lambda') + 2\lambda'\nu, \alpha'')n^h + \kappa m. \tag{124}$$

Thus, $\overline{w}^b(\alpha'') > 1$. Then by the Treasury bond market clearing condition we have

$$w^b(\alpha'')n = b - \underline{b} - \overline{n} - \mathcal{W}(2\nu(\rho + \lambda') + 2\lambda'\nu, \alpha'')n^h - \kappa m. \tag{125}$$

Let us solve for the threshold: $(\Theta'' - 2\lambda'\nu)/(\rho + \lambda') = 2\nu$. Thus,

$$\Theta'' = 2\nu(\rho + \lambda') + 2\lambda'\nu \tag{126}$$

and there is no discontinuity.

When a fire sale occurs, we have:

$$r^b(\alpha'') - r^d(\alpha'') = (\rho + 2\lambda')\nu + \chi\ell(\alpha''), \tag{127}$$

$$r^b(\alpha'') - r^p(\alpha'') = -(\rho + 2\lambda')\nu. \tag{128}$$

Furthermore, $r^d(\alpha'') - r^p(\alpha'') = -\chi\ell(\alpha'') - \vartheta^m(\alpha'') = -\chi\ell(\alpha'') - 2(\rho + 2\lambda')\nu$ and $r^d(\alpha'') = r^m(\alpha'') + \kappa\vartheta^m(\alpha'') - \chi\ell(\alpha'')$. Thus, $r^b(\alpha'') - r^m(\alpha'') = (1 + 2\kappa)(\rho + 2\lambda')\nu$ and $r^p(\alpha'') - r^m(\alpha'') = (2 + 2\kappa)(\rho + 2\lambda')\nu$.

Lemma A2. *There does not exist $\alpha' > \alpha^s$ such that $w^p(\alpha') < 0$. Thus, $\mathcal{A} = \emptyset$.*

Proof. Recall that from **??**, $w^p(\alpha^s) = 0$. Assume by contradiction that there exists some $\alpha' \in (\alpha^s, 1)$ such that $w^p(\alpha') < 0$. By Lemma A1, there are only two cases: $w^b(\alpha') = w^b(\alpha^s)$ and $w^b(\alpha') > w^b(\alpha^s)$.

- Case $w^b(\alpha') = w^b(\alpha^s)$. From the Treasury bond market clearing condition, we get that $\overline{w}^b(\alpha') = \overline{w}^b(\alpha^s)$. Since $w^k(\alpha') = w^k(\alpha^s)$, $w^m(\alpha') = w^m(\alpha^s)$, and $w^p(\alpha') < w^p(\alpha^s)$, then $w^d(\alpha') < w^d(\alpha^s)$ by the traditional bank balance sheet constraint. Then, by the deposit market clearing and the household balance sheet constraint, it must be that $w^{h,p}(\alpha') > w^{h,p}(\alpha^s)$.

  Since $w^{h,p}(\alpha') > w^{h,p}(\alpha^s)$ and $\alpha' > \alpha^s$, from the first-order condition of the household we get $r^{pt}(\alpha') - r^d(\alpha') > r^{pt}(\alpha^s) - r^d(\alpha^s)$. Since $r^{pt}(\alpha^s) - r^d(\alpha^s) \geq -\chi\ell(\alpha^s)$, $r^{pt}(\alpha') - r^d(\alpha') > -\chi\ell(\alpha^s)$. Note that $\ell(\alpha^s)n = (1 + \tau^h)n^h - f$ and $\ell(\alpha')n = w^d(\alpha')n + w^x(\alpha')n - [w^p(\alpha')]^- n = (1 + \tau^h)n^h - f - [w^p(\alpha')]^- n > (1 + \tau^h)n^h - f$. Therefore, $\ell(\alpha') > \ell(\alpha^s) > 0$ and $r^{pt}(\alpha') - r^d(\alpha') > -\chi\ell(\alpha^s) > -\chi\ell(\alpha')$. The last inequality is a contradiction with the traditional bank first-order conditions which require $r^{pt}(\alpha') - r^d(\alpha') = -\chi\ell(\alpha')$ if $w^p(\alpha') < 0$.

- Case $w^b(\alpha') > w^b(\alpha^s)$. By the market clearing condition for Treasury bonds, we need $\overline{w}^b(\alpha') < \overline{w}^b(\alpha^s)$. Then by the envelope conditions we get $r^p(\alpha') - r^d(\alpha') = 2\nu(\rho + \lambda') + 2\lambda'\nu + \chi\ell(\alpha') > 0$. This is a contradiction with the traditional bank first-order condition for bilateral repo which requires $r^p(\alpha') - r^d(\alpha') = 0$ if $w^p(\alpha') < 0$.

□

Lemma A3. *For all $\alpha' \in [\alpha^s, 1]$, $\ell(\alpha') = (1 + \tau^h)n^h - f$.*

Proof. From the triparti repo market clearing, we get $w^x(\alpha')n = w^{h,p}(\alpha')n^h - f$. From the deposit market clearing condition and the household balance sheet constraint, we obtain $w^d(\alpha')n = (1 + \tau^h)n^h - w^{h,p}(\alpha')n^h$. By definition, $\ell(\alpha') = w^d(\alpha') + w^x(\alpha') - [w^p(\alpha')]^-$. Since $w^p(\alpha') \geq 0$ by Lemma A2, then $\ell(\alpha') = w^x(\alpha') + w^d(\alpha') = (1 + \tau^h)n^h - f$. □

Lemma A4. *$\mathcal{S}$, $\mathcal{U}$, and $\mathcal{C}$ are intervals. Furthermore, $\mathcal{S}$, $\mathcal{U}$, and $\mathcal{C}$ form a partition of $(\alpha^s, 1)$ and $\mathcal{F}$ is an interval subset of $\mathcal{C}$. Finally, we can write*

$$\forall \alpha' \in \mathcal{S}, \alpha'' \in \mathcal{U} \cup \mathcal{C} : \alpha' < \alpha''; \tag{129}$$
$$\forall \alpha' \in \mathcal{U}, \alpha'' \in \mathcal{C} : \alpha' < \alpha''; \tag{130}$$
$$\forall \alpha' \in \mathcal{C} \setminus \mathcal{F}, \alpha'' \in \mathcal{F} : \alpha' < \alpha''. \tag{131}$$

Proof. We prove every statement sequentially.

- Statement 129. Consider arbitrary $\alpha' \in \mathcal{S}$ and $\alpha'' \in \mathcal{U}$. Then, $\Theta' = r^{pt}(\alpha') - r^d(\alpha') < 0$ and $\Theta'' = r^{pt}(\alpha'') - r^d(\alpha'') = 0$. Assume by way of contradiction that $\alpha'' \leq \alpha'$. We have that $w^p(\alpha') = 0$ and $w^p(\alpha'') \geq 0$. However, since there are no firesales in $\mathcal{S}$ and $\mathcal{U}$, $\overline{w}^p(\alpha') = \overline{w}^p(\alpha'') = \overline{w}^p(\alpha^s)$. Thus, from the bilateral repo market clearing condition, it must be that $w^x(\alpha'') \leq w^x(\alpha')$, and, from the triparty repo market clearing condition, it must be that $w^{h,p}(\alpha'') \leq w^{h,p}(\alpha')$. Using the household first-order condition for repo, we get

$$\Theta'' = 0 = \rho \frac{\beta}{1+\beta} \left(\frac{\alpha''}{1+\tau^h - w^{h,p}(\alpha'')} - \frac{1-\alpha''}{w^{h,p}(\alpha'')} \right) \tag{132}$$

  and

$$0 > \Theta' = \rho \frac{\beta}{1+\beta} \left(\frac{\alpha'}{1+\tau^h - w^{h,p}(\alpha')} - \frac{1-\alpha'}{w^{h,p}(\alpha')} \right), \tag{133}$$

  which together yield

$$\frac{\alpha''}{1+\tau^h - w^{h,p}(\alpha'')} - \frac{1-\alpha''}{w^{h,p}(\alpha'')} > \frac{\alpha'}{1+\tau^h - w^{h,p}(\alpha')} - \frac{1-\alpha'}{w^{h,p}(\alpha')}. \tag{134}$$

  Define a function

$$g(x, y) \equiv \frac{x}{1+\tau^h - y} - \frac{1-x}{y} \tag{135}$$

  and take the partial derivatives

$$\frac{\partial g}{\partial x} = \frac{1}{1+\tau^h - y} + \frac{1}{y}, \tag{136}$$
$$\frac{\partial g}{\partial y} = \frac{x}{(1+\tau^h - y)^2} + \frac{1-x}{y^2}. \tag{137}$$

  So long as $x \in (0, 1)$ and $y \in (0, 1+\tau^h)$, then both partial derivatives are positive. Since $w^{h,p} \in (0, 1+\tau^h)$ and $\alpha', \alpha'' \in (0, 1)$, then both derivatives are positive. Hence, $\alpha'' \leq \alpha'$ and $w^{h,p}(\alpha'') \leq w^{h,p}(\alpha')$ implies that $g(\alpha'', w^{h,p}(\alpha'')) \leq g(\alpha', w^{h,p}(\alpha'))$, a contradiction with inequality (134). Therefore $\alpha' < \alpha''$ for any $\alpha' \in \mathcal{S}$ and $\alpha'' \in \mathcal{U}$.

- Statement 130. Next consider arbitrary $\alpha' \in \mathcal{U}$ and $\alpha'' \in \mathcal{C} \setminus \mathcal{F}$. Then, $\Theta' = r^{pt}(\alpha') - r^d(\alpha') = 0$ and $\Theta'' = r^{pt}(\alpha'') - r^d(\alpha'') > 0$. Assume by way of contradiction that $\alpha'' < \alpha'$. We have that $w^p(\alpha') \in [0, \kappa m/n]$ and $w^p(\alpha'') = \kappa m/n$. However, $\overline{w}^p(\alpha') = \overline{w}^p(\alpha'') = \overline{w}^p(\alpha^s)$. By bilateral repo market clearing, then we must have that $w^x(\alpha'') \leq w^x(\alpha')$, and by triparty repo market clearing we must have that $w^{h,p}(\alpha'') \leq w^{h,p}(\alpha')$, which implies $\mathcal{W}(\Theta'', \alpha'') \leq \mathcal{W}(\Theta', \alpha')$. Using the household first-order condition for repo, we then have the following:

$$0 < \Theta'' = \rho \frac{\beta}{1+\beta} \left(\frac{\alpha''}{1+\tau^h - w^{h,p}(\alpha'')} - \frac{1-\alpha''}{w^{h,p}(\alpha'')} \right) \tag{138}$$

$$0 = \Theta' = \rho \frac{\beta}{1+\beta} \left(\frac{\alpha'}{1+\tau^h - w^{h,p}(\alpha')} - \frac{1-\alpha'}{w^{h,p}(\alpha')} \right), \tag{139}$$

which together yield

$$\frac{\alpha''}{1+\tau^h - w^{h,p}(\alpha'')} - \frac{1-\alpha''}{w^{h,p}(\alpha'')} > \frac{\alpha'}{1+\tau^h - w^{h,p}(\alpha')} - \frac{1-\alpha'}{w^{h,p}(\alpha')}. \tag{140}$$

Hence, since $w^{h,p}(\alpha'') < w^{h,p}(\alpha')$ and $\alpha' > \alpha''$, we have a contradiction. Therefore $\alpha' < \alpha''$ for any $\alpha' \in \mathcal{U}$ and $\alpha'' \in \mathcal{C} \setminus \mathcal{F}$.

- Statement 131. Finally, consider arbitrary $\alpha' \in \mathcal{C} \setminus \mathcal{F}$ and $\alpha'' \in \mathcal{F}$. Then, $\Theta' = r^{pt}(\alpha') - r^d(\alpha') > 0$ and $\Theta'' = r^{pt}(\alpha'') - r^d(\alpha'') > 0$. Assume by way of contradiction that $\alpha'' < \alpha'$. We have that $w^p(\alpha') = w^p(\alpha'') = \kappa m/n$. Given that there is a firesale in $\mathcal{F}$, $\overline{w}^p(\alpha') = \overline{w}^p(\alpha^s) > \overline{w}^p(\alpha'')$. Thus, from the bilateral repo market clearing condition, it must be that $w^x(\alpha'') < w^x(\alpha')$, and, from the triparty repo market clearing condition, it must be that $w^{h,p}(\alpha'') < w^{h,p}(\alpha')$. Using the household first-order condition for repo, we then have the following:

$$0 < \Theta'' = \rho \frac{\beta}{1+\beta} \left(\frac{\alpha''}{1+\tau^h - w^{h,p}(\alpha'')} - \frac{1-\alpha''}{w^{h,p}(\alpha'')} \right) \tag{141}$$

$$0 < \Theta' = \rho \frac{\beta}{1+\beta} \left(\frac{\alpha'}{1+\tau^h - w^{h,p}(\alpha')} - \frac{1-\alpha'}{w^{h,p}(\alpha')} \right) \tag{142}$$

  Since $w^{h,p}(\alpha'') < w^{h,p}(\alpha')$ and $\alpha' > \alpha''$, $\Theta'' < \Theta'$. However, by the envelope conditions, it must be that $\Theta'' = 2\nu(\rho + \lambda') + 2\lambda'\nu$ and $\Theta' = (\rho + \lambda')(\theta(\alpha') - \overline{\theta}(\alpha')) + 2\lambda'\nu$ with $\theta(\alpha') - \overline{\theta}(\alpha') \leq 2\nu$, which implies $\Theta' \leq \Theta''$, and we have a contradiction. Therefore $\alpha' < \alpha''$ for any $\alpha' \in \mathcal{C} \setminus \mathcal{F}$ and $\alpha'' \in \mathcal{F}$.

□

Given Lemma A4 and the definition of the partitions, we can define the thresholds of each partition as follows:

$$\mathcal{S} = [\alpha^s, \alpha^{\mathcal{U}}),\ \mathcal{U} = [\alpha^{\mathcal{U}}, \alpha^{\mathcal{C}}],\ \mathcal{C} = (\alpha^{\mathcal{C}}, 1),\ \mathcal{F} = (\alpha^{\mathcal{F}}, 1) \tag{143}$$

and $\alpha^s \leq \alpha^{\mathcal{U}} \leq \alpha^{\mathcal{C}} \leq \alpha^{\mathcal{F}}$. Recall the following definitions $\Theta^s \equiv r^{pt}(\alpha^s) - r^d(\alpha^s)$ and $\Theta(\alpha') \equiv r^{pt}(\alpha') - r^d(\alpha')$. Define $\mathcal{W}^s \equiv \mathcal{W}(\Theta^s, \alpha^s) = w^{h,p}(\alpha^s)$ and $\mathcal{W}^{\mathcal{F}} \equiv \mathcal{W}(\Theta^{\mathcal{F}}, \alpha^{\mathcal{F}}) = w^{h,p}(\alpha^{\mathcal{F}})$ where $\Theta^{\mathcal{F}} \equiv r^{pt}(\alpha^{\mathcal{F}}) - r^d(\alpha^{\mathcal{F}})$ and $\mathcal{W}(\Theta, \alpha)$ is defined in equation (94).

Lemma A5. *The thresholds are defined by*

$$\alpha^{\mathcal{U}} = \frac{1+\tau^h - \mathcal{W}^s}{1+\tau^h}, \tag{144}$$

$$\alpha^{\mathcal{C}} = \frac{1+\tau^h - \mathcal{W}^{\mathcal{F}}}{1+\tau^h},. \tag{145}$$

Furthermore,

$$\mathcal{W}^{\mathcal{F}} = \mathcal{W}^s - \frac{\kappa m}{n^h}, \tag{146}$$

$$\Theta^{\mathcal{F}} = 2\nu(\rho + \lambda') + 2\lambda'\nu. \tag{147}$$

Proof. Given the continuity of $\mathcal{W}$ and by definition of $\mathcal{S}$ and $\mathcal{U}$, $\alpha^{\mathcal{U}}$ is such that $\Theta(\alpha^{\mathcal{U}}) = 0$ and $\mathcal{W}(\Theta(\alpha^{\mathcal{U}}), \alpha^{\mathcal{U}}) = \mathcal{W}^s$. These two conditions together yield equation (144).

Similarly, from equation (118), $\alpha^{\mathcal{C}}$ is such that $\Theta(\alpha^{\mathcal{C}}) = 0$ and $\mathcal{W}^s - \mathcal{W}(\Theta(\alpha^{\mathcal{C}}), \alpha^{\mathcal{C}}) = \kappa m/n^h$. These two conditions together yield equation (145).

The other statements follow from equation (126). □

Lemma A6.

$$\frac{\partial \alpha^{\mathcal{F}}}{\partial \lambda} > 0 \text{ and } \frac{\partial \overline{w}^b(\alpha^s)}{\partial \lambda} < 0$$

Proof. Given Lemma A4, we get that for $\alpha' \in [\alpha^s, \alpha^{\mathcal{F}}]$, $\overline{w}^b(\alpha') = \overline{w}^b(\alpha^s)$ and $w^b(\alpha') = w^b(\alpha^s)$, and for $\alpha'' \in [\alpha^{\mathcal{F}}, 1]$, $\overline{w}^b(\alpha'') < \overline{w}^b(\alpha^s)$ and $w^b(\alpha'') > w^b(\alpha^s)$. Note that $f(\alpha) = 1/(1-\alpha^s)$. Thus, we can rewrite equation (92) as

$$\begin{aligned} r^{pt}(\alpha^s) - r^d(\alpha^s) = -2\nu(\rho+\lambda) - 2\lambda\nu\frac{1-\alpha^{\mathcal{F}}}{1-\alpha^s} \\ - \frac{\lambda}{1-\alpha^s}\int_{\alpha^s}^{\alpha^{\mathcal{F}}} (\theta(\alpha') - \overline{\theta}(\alpha'))d\alpha'. \end{aligned} \tag{148}$$

Similarly, we can rewrite equation (105) as

$$r^{pt}(\alpha') - r^d(\alpha') = 2\lambda'\nu + (\rho+\lambda')(\theta(\alpha') - \overline{\theta}(\alpha')). \tag{149}$$

Using the above notation, we can combine the previous expressions:

$$\begin{aligned} \Theta^s = -2\nu(\rho+\lambda) - 2\lambda\nu\frac{1-\alpha^{\mathcal{F}}}{1-\alpha^s} + \frac{2\lambda\lambda'\nu(\alpha^{\mathcal{F}}-\alpha^s)}{(1-\alpha^s)(\rho+\lambda')} \\ - \frac{\lambda}{(1-\alpha^s)(\rho+\lambda')}\int_{\alpha^s}^{\alpha^{\mathcal{F}}} \Theta' d\alpha'. \end{aligned} \tag{150}$$

We now take the derivative with respect to λ of both sides of the following three equations, leading to a system of three equations and three unknowns.

$$\Theta^s = \frac{\rho\beta}{1+\beta}\left(\frac{\alpha^s}{1+\tau^h-\mathcal{W}^s} - \frac{1-\alpha^s}{\mathcal{W}^s}\right) \tag{151}$$

$$\Theta^{\mathcal{F}} = \frac{\rho\beta}{1+\beta}\left(\frac{\alpha^{\mathcal{F}}}{1+\tau^h-\mathcal{W}^{\mathcal{F}}} - \frac{1-\alpha^{\mathcal{F}}}{1+\tau^h-\mathcal{W}^{\mathcal{F}}}\right) = 2\lambda'\nu + 2\nu(\rho+\lambda') \tag{152}$$

$$\Theta^s = -2\nu(\rho+\lambda) - 2\lambda\nu\frac{1-\alpha^{\mathcal{F}}}{1-\alpha^s} + \frac{2\lambda\lambda'\nu(\alpha^{\mathcal{F}}-\alpha^s)}{(1-\alpha^s)(\rho+\lambda')} - \frac{\lambda}{(1-\alpha^s)(\rho+\lambda')}\int_{\alpha^s}^{\alpha^{\mathcal{F}}} \Theta' d\alpha' \tag{153}$$

The first equation is simply the household first order condition in the normal state. The second equation is the household first-order condition at $\alpha' = \alpha^{\mathcal{F}}$. And we derived the third equation in equation (150).

Taking derivatives, we end up with the following system of equations:

$$
\begin{aligned}
\frac{\partial \Theta^s}{\partial \lambda} &= \frac{\rho\beta}{1+\beta}\left(\frac{\alpha^s}{(1+\tau^h-\mathcal{W}^s)^2}+\frac{1-\alpha^s}{(\mathcal{W}^s)^2}\right)\frac{\partial \mathcal{W}^s}{\partial \lambda} \\
0 &= \frac{\rho\beta}{1+\beta}\left[\left(\frac{\alpha^{\mathcal{F}}}{(1+\tau^h-\mathcal{W}^{\mathcal{F}})^2}+\frac{1-\alpha^{\mathcal{F}}}{(\mathcal{W}^{\mathcal{F}})^2}\right)\frac{\partial \mathcal{W}^{\mathcal{F}}}{\partial \lambda}+\left(\frac{1}{1+\tau^h-\mathcal{W}^{\mathcal{F}}}+\frac{1}{\mathcal{W}^{\mathcal{F}}}\right)\frac{\partial \alpha^{\mathcal{F}}}{\partial \lambda}\right] \\
\frac{\partial \Theta^s}{\partial \lambda} &= -2\nu-2\nu\frac{1-\alpha^{\mathcal{F}}}{1-\alpha^s}+\frac{2\lambda\nu}{1-\alpha^s}\frac{\partial \alpha^{\mathcal{F}}}{\partial \lambda}+\frac{2\lambda'\nu(\alpha^{\mathcal{F}}-\alpha^s)}{(1-\alpha^s)(\rho+\lambda')}+\frac{2\lambda\lambda'\nu}{(1-\alpha^s)(\rho+\lambda')}\frac{\partial \alpha^{\mathcal{F}}}{\partial \lambda} \\
&\quad -\frac{1}{(1-\alpha^s)(\rho+\lambda')}\frac{\rho\beta}{1+\beta}\left(\frac{(\alpha^{\mathcal{U}}-\alpha^s)(2\mathcal{W}^s+(1+\tau^h)(\alpha^s+\alpha^{\mathcal{U}}-2))}{2\mathcal{W}^s(1+\tau^h-\mathcal{W}^s)}\right. \\
&\qquad \left. +\frac{(\alpha^{\mathcal{F}}-\alpha^{\mathcal{C}})(2\mathcal{W}^{\mathcal{F}}+(1+\tau^h)(\alpha^{\mathcal{F}}+\alpha^{\mathcal{C}}-2))}{2\mathcal{W}^{\mathcal{F}}(1+\tau^h-\mathcal{W}^{\mathcal{F}})}\right) \\
&\quad -\frac{\lambda}{(1-\alpha^s)(\rho+\lambda')}\frac{\rho\beta}{1+\beta}\left(-\frac{((\mathcal{W}^s+(1+\tau^h)(\alpha^s-1))(\mathcal{W}^s(2\alpha^s-1))}{2(\mathcal{W}^s)^2(1+\tau^h-\mathcal{W}^s)^2}\frac{\partial \mathcal{W}^s}{\partial \lambda}\right. \\
&\qquad +\frac{((\mathcal{W}^s+(1+\tau^h)(\alpha^s-1))(1+\tau^h)(\alpha^s-1)}{2(\mathcal{W}^s)^2(1+\tau^h-\mathcal{W}^s)^2}\frac{\partial \mathcal{W}^s}{\partial \lambda} \\
&\qquad \left. +\frac{1}{2}\left(\frac{(\alpha^{\mathcal{F}})^2}{(1+\tau^h-\mathcal{W}^{\mathcal{F}})^2}-\frac{(\alpha^{\mathcal{F}}-1)^2}{(\mathcal{W}^{\mathcal{F}})^2}\right)\frac{\partial \mathcal{W}^{\mathcal{F}}}{\partial \lambda}+\left(\frac{\alpha^{\mathcal{F}}}{1+\tau^h-\mathcal{W}^{\mathcal{F}}}-\frac{1-\alpha^{\mathcal{F}}}{\mathcal{W}^{\mathcal{F}}}\right)\frac{\partial \alpha^{\mathcal{F}}}{\partial \lambda}\right)
\end{aligned}
\tag{154}
$$

This is a system with only three unknowns $\frac{\partial \mathcal{W}^s}{\partial \lambda}$, $\frac{\partial \Theta^s}{\partial \lambda}$, and $\frac{\partial \alpha^{\mathcal{F}}}{\partial \lambda}$, because $\frac{\partial \mathcal{W}^{\mathcal{F}}}{\partial \lambda} = \frac{\partial \mathcal{W}^s}{\partial \lambda}$, which derives from $\mathcal{W}^{\mathcal{F}} = \mathcal{W}^s - \frac{\kappa m}{n^h}$. For tractability, we introduce a number of intermediate values as follows:

$$\begin{aligned}
A &\equiv \frac{\alpha^s}{(1+\tau^h-\mathcal{W}^s)^2}+\frac{1-\alpha^s}{(\mathcal{W}^s)^2}\\
B &\equiv \frac{\rho\beta}{1+\beta}\\
C &\equiv \frac{\alpha^{\mathcal{F}}}{(1+\tau^h-\mathcal{W}^{\mathcal{F}})^2}+\frac{1-\alpha^{\mathcal{F}}}{(\mathcal{W}^{\mathcal{F}})^2}\\
D &\equiv \frac{1}{1+\tau^h-\mathcal{W}^{\mathcal{F}}}+\frac{1}{\mathcal{W}^{\mathcal{F}}}\\
E &\equiv \frac{2\lambda\nu}{1-\alpha^s}\\
F &\equiv -2\nu\left(1+\frac{1-\alpha^{\mathcal{F}}}{1-\alpha^s}\right)+\frac{2\lambda'\nu(\alpha^{\mathcal{F}}-\alpha^s)}{(1-\alpha^s)(\rho+\lambda')}\\
&\quad -\frac{\rho\beta}{(1-\alpha^s)(\rho+\lambda')(1+\beta)}\left(\frac{(\alpha^{\mathcal{U}}-\alpha^s)(2\mathcal{W}^s+(1+\tau^h)(\alpha^s+\alpha^{\mathcal{U}}-2))}{2\mathcal{W}^s(1+\tau^h-\mathcal{W}^s)}\right.\\
&\qquad\left.+\frac{(\alpha^{\mathcal{F}}-\alpha^{\mathcal{C}})(2\mathcal{W}^{\mathcal{F}}+(1+\tau^h)(\alpha^{\mathcal{F}}+\alpha^{\mathcal{C}}-2))}{2\mathcal{W}^{\mathcal{F}}(1+\tau^h-\mathcal{W}^{\mathcal{F}})}\right)\\
G &\equiv (\rho+\lambda')(1-\alpha^s)\\
H &\equiv \frac{\lambda B}{G}\frac{(\mathcal{W}^s+(1+\tau^h)(\alpha^s-1))(\mathcal{W}^s(2\alpha^s-1)-(1+\tau^h)(\alpha^s-1))}{2(\mathcal{W}^s)^2(1+\tau^h-\mathcal{W}^s)^2}\\
I &\equiv -\frac{\lambda B}{2G}\left(\frac{(\alpha^{\mathcal{F}})^2}{(1+\tau^h-\mathcal{W}^{\mathcal{F}})^2}-\frac{(\alpha^{\mathcal{F}}-1)^2}{(\mathcal{W}^{\mathcal{F}})^2}\right)\\
J &\equiv -\frac{\lambda B}{G}\left(\frac{\alpha^{\mathcal{F}}}{1+\tau^h-\mathcal{W}^{\mathcal{F}}}-\frac{1-\alpha^{\mathcal{F}}}{\mathcal{W}^{\mathcal{F}}}\right)\\
K &\equiv \frac{2\lambda\lambda'\nu}{G}
\end{aligned}$$

Since $\alpha^s \in [0,1]$, $A > 0$. Since $\rho, \beta > 0$, then $B > 0$. Assuming $0 < \rho < 1$ and $\beta \in [0,1]$, then $B \in (0,1)$. Since $\mathcal{F} \neq \emptyset$, $\alpha^{\mathcal{F}} \in (0,1)$, which implies $C > 0$. Furthermore, since $w^{h,p}(\alpha) \in (0, 1+\tau^h)$ for any α, $D > 0$. In addition, $\alpha^s \in (0,1)$ and $\lambda, \lambda', \nu > 0$ imply that $E, G, K > 0$. Also, we can simplify J as $J = -\lambda\Theta^{\mathcal{F}}/G$. Since $\Theta^{\mathcal{F}} = 2\nu(\rho+\lambda') + 2\lambda'\nu > 0$ and $\lambda, G > 0$, $J < 0$.

Next, we examine I. We can use a simple algebraic factorization as follows:

$$\begin{aligned}
I &= -\frac{\lambda B}{2G}\left(\frac{(\alpha^{\mathcal{F}})^2}{(1+\tau^h-\mathcal{W}^{\mathcal{F}})^2}-\frac{(1-\alpha^{\mathcal{F}})^2}{(\mathcal{W}^{\mathcal{F}})^2}\right)\\
&= -\frac{\lambda B}{2G}\left(\frac{\alpha^{\mathcal{F}}}{1+\tau^h-\mathcal{W}^{\mathcal{F}}}-\frac{1-\alpha^{\mathcal{F}}}{\mathcal{W}^{\mathcal{F}}}\right)\left(\frac{\alpha^{\mathcal{F}}}{1+\tau^h-\mathcal{W}^{\mathcal{F}}}+\frac{1-\alpha^{\mathcal{F}}}{\mathcal{W}^{\mathcal{F}}}\right)\\
&= -\frac{\lambda\Theta^{\mathcal{F}}}{2G}\left(\frac{\alpha^{\mathcal{F}}}{1+\tau^h-\mathcal{W}^{\mathcal{F}}}+\frac{1-\alpha^{\mathcal{F}}}{\mathcal{W}^{\mathcal{F}}}\right).
\end{aligned}\tag{155}$$

Since $\lambda, G, \Theta^{\mathcal{F}} > 0$, $\mathcal{W}^{\mathcal{F}} \in (0, 1+\tau^h)$ and $\alpha^{\mathcal{F}} \in (0,1)$, we have $I < 0$.

For H, we can also use an algebraic factorization as follows:

$$
\begin{aligned}
H &\equiv \frac{\lambda B}{G} \frac{(\mathcal{W}^s + (1+\tau^h)(\alpha^s - 1))(\mathcal{W}^s(2\alpha^s - 1) - (1+\tau^h)(\alpha^s - 1))}{2(\mathcal{W}^s)^2(1+\tau^h - \mathcal{W}^s)^2} \\
&= \frac{\lambda B}{2G} \frac{(\alpha^s\mathcal{W}^s + (1+\tau^h - \mathcal{W})(\alpha^s - 1))(\alpha^s\mathcal{W}^s - (1+\tau^h - \mathcal{W})(\alpha^s - 1))}{(\mathcal{W}^s)^2(1+\tau^h - \mathcal{W}^s)^2} \\
&= \frac{\lambda B}{2G} \left(\frac{\alpha^s}{1+\tau^h - \mathcal{W}^s} - \frac{1-\alpha^s}{\mathcal{W}^s} \right) \left(\frac{\alpha^s}{1+\tau^h - \mathcal{W}^s} + \frac{1-\alpha^s}{\mathcal{W}^s} \right). \qquad (156)
\end{aligned}
$$

Since $\mathcal{W}^s \in (0, 1+\tau^h)$, $0 < \alpha^s < \alpha^{\mathcal{U}} < 1$, and $\alpha^{\mathcal{U}}/(1+\tau^h - \mathcal{W}^s) - (1-\alpha^{\mathcal{U}})/\mathcal{W}^s = 0$, then $\alpha^s/(1+\tau^h - \mathcal{W}^s) - (1-\alpha^s)/\mathcal{W}^s < 0$. Together with $\lambda, G > 0$, and $\alpha^s \in (0,1)$, we get $H < 0$.

Finally, we turn to look at F. F can be simplified as

$$
\begin{aligned}
F &= \frac{1}{G}\left(\Theta^{\mathcal{F}}(\alpha^{\mathcal{F}} - 1) + 2\nu\rho(\alpha^s - 1)\right) - \frac{1}{2G}\left(\Theta^s(\alpha^{\mathcal{U}} - \alpha^s) + \Theta^{\mathcal{F}}(\alpha^{\mathcal{F}} - \alpha^{\mathcal{C}})\right) \\
&= \frac{1}{2G}\left(\Theta^{\mathcal{F}}(\alpha^{\mathcal{F}} + \alpha^{\mathcal{C}} - 2) + \Theta^s(\alpha^s - \alpha^{\mathcal{U}}) + 4\nu\rho(\alpha^s - 1)\right). \qquad (157)
\end{aligned}
$$

Note that Θ^s can also be expressed as

$$
\begin{aligned}
\Theta^s = &-2\nu(\rho+\lambda) - 2\lambda\nu\frac{1-\alpha^{\mathcal{F}}}{1-\alpha^s} + \frac{2\lambda\lambda'\nu(\alpha^{\mathcal{F}} - \alpha^s)}{(1-\alpha^s)(\rho+\lambda')} \\
&- \frac{\lambda}{(1-\alpha^s)(\rho+\lambda')}\frac{\rho\beta}{1+\beta}\left(\frac{(\alpha^{\mathcal{U}} - \alpha^s)(2\mathcal{W}^s + (1+\tau^h)(\alpha^s + \alpha^{\mathcal{U}} - 2))}{2\mathcal{W}^s(1+\tau^h - \mathcal{W}^s)}\right. \\
&\left. + \frac{(\alpha^{\mathcal{F}} - \alpha^{\mathcal{C}})(2\mathcal{W}^{\mathcal{F}} + (1+\tau^h)(\alpha^{\mathcal{F}} + \alpha^{\mathcal{C}} - 2))}{2\mathcal{W}^{\mathcal{F}}(1+\tau^h - \mathcal{W}^{\mathcal{F}})}\right). \qquad (158)
\end{aligned}
$$

Thus,

$$
F = \frac{\Theta^s}{\lambda} + \frac{2\rho\nu}{\lambda}. \qquad (159)
$$

If we assume $F \geq 0$, then $\Theta^s \geq -2\rho\nu$ must hold. Since $\alpha^s \leq \alpha^{\mathcal{U}}$, it implies

$$
\begin{aligned}
F &= \frac{1}{2G}\left(\Theta^{\mathcal{F}}(\alpha^{\mathcal{F}} + \alpha^{\mathcal{C}} - 2) + \Theta^s(\alpha^s - \alpha^{\mathcal{U}}) + 4\nu\rho(\alpha^s - 1)\right) \\
&\leq \frac{1}{2G}\left(\Theta^{\mathcal{F}}(\alpha^{\mathcal{F}} + \alpha^{\mathcal{C}} - 2) + 2\rho\nu(\alpha^{\mathcal{U}} - 1) + 2\nu\rho(\alpha^s - 1)\right) < 0. \qquad (160)
\end{aligned}
$$

The last inequality stems from $\alpha^s, \alpha^{\mathcal{U}}, \alpha^{\mathcal{C}}, \alpha^{\mathcal{F}} \in (0,1)$ and $\Theta^{\mathcal{F}} > 0$. This contradicts with the initial assumption $F \geq 0$. Hence, $F < 0$.

To recap, we have

$$
F, H, I, J < 0 \qquad (161)
$$

$$
A, B, C, D, E, G, K > 0 \qquad (162)
$$

Using these intermediate variables, we can rewrite the system of three equations and three unknowns as

$$\frac{\partial \Theta^s}{\partial \lambda} = AB\frac{\partial \mathcal{W}^s}{\partial \lambda}, \tag{163}$$

$$0 = C\frac{\partial \mathcal{W}^s}{\partial \lambda} + D\frac{\partial \alpha^{\mathcal{F}}}{\partial \lambda}, \tag{164}$$

$$\frac{\partial \Theta^s}{\partial \lambda} = F + (E+K+J)\frac{\partial \alpha^{\mathcal{F}}}{\partial \lambda} + (H+I)\frac{\partial \mathcal{W}^s}{\partial \lambda}. \tag{165}$$

We can solve $\frac{\partial \alpha^{\mathcal{F}}}{\partial \lambda}$ as

$$\frac{\partial \alpha^{\mathcal{F}}}{\partial \lambda} = \frac{F}{(H+I-AB)(D/C)-(E+K+J)}. \tag{166}$$

The numerator, F, is negative. For the denominator, since $H, I < 0$ and $A, B, C, D > 0$, we have $(H+I-AB)(D/C) < 0$. Finally,

$$E + K + J = \frac{1}{G}\left(2\lambda\nu(\rho+\lambda') + 2\lambda\lambda'\nu - \lambda(2\nu(\rho+\lambda') + 2\lambda'\nu)\right) = 0$$

Hence, the denominator is negative, and thereby $\frac{\partial \alpha^{\mathcal{F}}}{\partial \lambda} > 0$.

Since $0 = C\frac{\partial \mathcal{W}^s}{\partial \lambda} + D\frac{\partial \alpha^{\mathcal{F}}}{\partial \lambda}$ and $C, D, \frac{\partial \alpha^{\mathcal{F}}}{\partial \lambda} > 0$, then $\frac{\partial \mathcal{W}^s}{\partial \lambda} = \frac{\partial w^{h,p}(\alpha^s)}{\partial \lambda} < 0$. By assumption, $w^p(\alpha^s) = 0$. By bilateral repo market clearing we have $w^p(\alpha^s)n + w^x(\alpha^s)n + f = \overline{w}^p(\alpha^s)\overline{n}$. By triparty repo market clearing we have $w^{h,p}(\alpha^s)n^h = f + w^x(\alpha^s)n$. Combining, we arrive at $w^{h,p}(\alpha^s)n^h = \overline{w}^p(\alpha^s)\overline{n}$. Noting that the shadow bank balance sheet constraint we have $\overline{w}^b(\alpha^s) = 1 + \overline{w}^p(\alpha^s)$, then we get

$$\overline{n}\frac{\partial}{\partial \lambda}\overline{w}^b(\alpha^s) = \frac{\partial}{\partial \lambda}(\overline{n} + \overline{w}^p(\alpha^s)\overline{n}) = \frac{\partial}{\partial \lambda}(w^{h,p}(\alpha^s)n^h) = n^h\frac{\partial \mathcal{W}^s}{\partial \lambda}. \tag{167}$$

Since $n, n^h, \overline{n} > 0$ and $\frac{\partial \mathcal{W}^s}{\partial \lambda} < 0$, it must be that $\frac{\partial \overline{w}^b(\alpha^s)}{\partial \lambda} < 0$. □

Lemma A7.

$$\frac{\partial}{\partial \lambda}\mathbb{E}[r^p(\alpha') - r^m(\alpha')] < 0$$

Proof. Using the traditional bank first order conditions, we can write

$$\begin{aligned}\mathbb{E}[r^p(\alpha') - r^m(\alpha')] &= \mathbb{E}[r^p(\alpha') - r^d(\alpha')] - \mathbb{E}[r^m(\alpha') - r^d(\alpha')] \\ &= \frac{1}{1-\alpha^s}\left(\int_{\alpha^s}^1 \Theta(\alpha')d\alpha' + \kappa\int_{\alpha^s}^1 \vartheta^m(\alpha')d\alpha'\right).\end{aligned} \tag{168}$$

Since $\kappa m > 0$, $\vartheta^m(\alpha') > 0$ only if $w^p(\alpha') > 0$. Thus, from the traditional bank first-order conditions, $\vartheta^m(\alpha') > 0$ only $r^p(\alpha') - r^m(\alpha') > 0$—that is, only if $\alpha \in (\alpha^s, \alpha^{\mathcal{C}})$. Furthermore, when $\vartheta^m > 0$, then $r^{pt}(\alpha') - r^d(\alpha') = \vartheta^m(\alpha')$. Thus,

$$\mathbb{E}[r^p(\alpha') - r^m(\alpha')] = \frac{1}{1-\alpha^s}\left(\int_{\alpha^s}^1 \Theta(\alpha')d\alpha' + \kappa\int_{\alpha^{\mathcal{C}}}^1 \Theta(\alpha')d\alpha'\right). \tag{169}$$

Since $\Theta(\alpha') = 0$ on $[\alpha^{\mathcal{U}}, \alpha^{\mathcal{C}}]$, $\Theta(\alpha') = \frac{\rho\beta}{1+\beta}\left(\frac{\alpha'}{1+\tau^h-\mathcal{W}^s} - \frac{1-\alpha'}{\mathcal{W}^s}\right)$ for $\alpha' \in (\alpha^s, \alpha^{\mathcal{U}})$, $\Theta(\alpha') = \frac{\rho\beta}{1+\beta}\left(\frac{\alpha'}{1+\tau^h-\mathcal{W}^{\mathcal{F}}} - \frac{1-\alpha'}{\mathcal{W}^{\mathcal{F}}}\right)$ for $\alpha' \in (\alpha^{\mathcal{C}}, \alpha^{\mathcal{F}}]$, and $\Theta(\alpha') = 2\nu(\rho+\lambda') + 2\lambda'\nu$ for $\alpha' \in (\alpha^{\mathcal{F}}, 1)$, we get

$$\mathbb{E}[r^p(\alpha') - r^m(\alpha')] = \frac{1}{1-\alpha^s}\frac{\rho\beta}{1+\beta}\left[\frac{(\alpha^{\mathcal{U}}-\alpha^s)(2\mathcal{W}^s+(1+\tau^h)(\alpha^{\mathcal{U}}+\alpha^s-2)}{2\mathcal{W}^s(1+\tau^h-\mathcal{W}^s)}\right.$$
$$\left.+(1+\kappa)\frac{(\alpha^{\mathcal{F}}-\alpha^{\mathcal{C}})(2\mathcal{W}^{\mathcal{F}}+(1+\tau^h)(\alpha^{\mathcal{F}}+\alpha^{\mathcal{C}}-2))}{2\mathcal{W}^{\mathcal{F}}(1+\tau^h-\mathcal{W}^{\mathcal{F}})}\right]$$
$$+\frac{1+\kappa}{1-\alpha^s}(1-\alpha^{\mathcal{F}})(2\nu(\rho+\lambda')+2\lambda'\nu). \tag{170}$$

Taking the partial derivative with respect to λ, using the definition of intermediates defined in Lemma A6, noting that $\frac{\partial\mathcal{W}^s}{\partial\lambda} = \frac{\partial\mathcal{W}^{\mathcal{F}}}{\partial\lambda}$, we get

$$\frac{\partial}{\partial\lambda}\mathbb{E}[r^p(\alpha') - r^m(\alpha')] = -\frac{\rho+\lambda'}{\lambda}(H+(1+\kappa)I)\frac{\partial\mathcal{W}^s}{\partial\lambda} - (1+\kappa)\left(\frac{\Theta^{\mathcal{F}}}{1-\alpha^s}+\frac{\rho+\lambda'}{\lambda}J\right)\frac{\partial\alpha^{\mathcal{F}}}{\partial\lambda}. \tag{171}$$

Given that $\frac{\rho\beta}{1+\beta}\left(\frac{\alpha^{\mathcal{F}}}{1+\tau^h-\mathcal{W}^{\mathcal{F}}} - \frac{1-\alpha^{\mathcal{F}}}{\mathcal{W}^{\mathcal{F}}}\right) = \Theta^{\mathcal{F}}$, we get

$$\frac{1}{1-\alpha^s}\Theta^{\mathcal{F}} + \frac{\rho+\lambda'}{\lambda}J = 0 \tag{172}$$

and

$$\frac{\partial}{\partial\lambda}\mathbb{E}[r^p(\alpha') - r^m(\alpha')] = -\frac{\rho+\lambda'}{\lambda}(H+(1+\kappa)I)\frac{\partial\mathcal{W}^s}{\partial\lambda}. \tag{173}$$

Since $\frac{\partial\mathcal{W}^s}{\partial\lambda} < 0$, $\kappa, \rho, \lambda'\,\lambda > 0$ and $H, I < 0$, then

$$\frac{\partial}{\partial\lambda}\mathbb{E}[r^p(\alpha') - r^m(\alpha')] < 0. \tag{174}$$

□

The proof of Proposition 1 then directly follows from Lemmas A6 and A7.

In Section 5, we solve for equilibria assuming that the state variables are constant over time. Implicitly, we assume that α_t is constant over time and the transaction cost κ is sufficiently high to keep the allocation of Treasury bonds constant over time ($w^b_t = w^b_{t\text{-}} = w^b$ and $\overline{w}^b_t = \overline{w}^b_{t\text{-}} = \overline{w}^b$). Thus, for ease of notation, we drop the time subscript.

B.1 Proof of Lemma 2

Liquidity services $h(w^{h,p}, w^{h,d}, \alpha)$, given the budget constraint (17), are maximized when $w^{h,p} = (1-\alpha)(1+\tau^h)$ and $w^{h,d} = \alpha(1+\tau^h)$. Given households' first-order condition for triparty repo in Equation 23, this occurs if and only if $r^{pt} = r^d$. Combining Equations 19 and 22 obtains $r^p - r^k = r^{pt} - r^d$. Thus, $r^{pt} = r^d$ if and only if $r^p = r^k$.

Assume that $r^p = r^k$. If $\vartheta^m > 0$, then $w^p = \kappa w^m > 0$ since $m > 0$ and $r^p > r^k$ given the first-order condition for bilateral repo of traditional banks in Equation 21. Thus, if $r^p = r^k$, then $\vartheta^m = 0$ and $r^k = r^m$ and $r^p = r^m$. Thus, $\mathcal{M}(x) \in \mathcal{U}$.

Assume that $\mathcal{M}(x) \in \mathcal{U}$. Then $\vartheta^m = 0$. Assume the contrary: $\vartheta^m > 0$. Then $w^p = \kappa w^m > 0$ since $m > 0$ and $r^k = r^m + \kappa\vartheta^m = r^p - \vartheta^m$. Thus, $r^m < r^p$, a contradiction. Thus, $r^k = r^m = r^p$.

Therefore, liquidity services are at the optimum if and only if $\mathcal{M}(x) \in \mathcal{U}$.

Lemma A8. *Combining all the market clearing conditions, we get*

$$w^{h,p}n^h = b - \underline{b} - w^b n - \overline{n} - w^p n.$$

Proof. From the Treasury bond market clearing condition, we have $\overline{w}^b\overline{n} = b - \underline{b} - w^b n$. Then, substituting $\overline{w}^p$ by the shadow bank balance sheet constraint, we get $\overline{w}^p\overline{n} = b - \underline{b} - w^b n - \overline{n}$. From the bilateral repo market clearing condition, we get $w^x n + w^p n + f = b - \underline{b} - w^b n - \overline{n}$. Combined with the triparty repo market clearing condition, we get $w^{h,p}n^h + w^p n = b - \underline{b} - w^b n - \overline{n}$. □

Lemma A9. *In the absence of repo and reverse repo facilities, the IL constraint binds ($\vartheta^m > 0$) if and only if $b - \underline{b} - w^b n > \overline{n} + (1-\alpha)(1+\tau^h)n^h + w^p n$.*

Proof. Assume that $b - \underline{b} - w^b n > \overline{n} + (1-\alpha)(1+\tau^h)n^h + \kappa m$. Using Lemma A8, we get $w^{h,p}n^h > (1-\alpha)(1+\tau^h)n^h + \kappa m - w^p n \geq (1-\alpha)(1+\tau^h)n^h$. Then, by Lemma 2, $r^{pt} > r^d$ and by Equations 19, 20, 21, and 22, $\vartheta^m > 0$.

Assume that $\vartheta^m > 0$. Then, $w^p = \kappa w^m > 0$ and by Equations 19, 21, and 22, $r^{pt} > r^d$. Thus, by Lemma 2, $w^{h,p}n^h > (1-\alpha)(1+\tau^h)n^h$. Using Lemma A8, we get $b - \underline{b} - w^b n - \overline{n} - w^p n > (1-\alpha)(1+\tau^h)n^h$. Thus, $b - \underline{b} - w^b n > \overline{n} + (1-\alpha)(1+\tau^h)n^h + w^p n$. □

Lemma A10. *In the absence of repo and reverse repo facilities, $r^{pt} < r^d$ if and only if $b - \underline{b} - w^b n < \overline{n} + (1-\alpha)(1+\tau^h)n^h$.*

Proof. Assume that $r^{pt} < r^d$. Thus, by Lemma 2, $w^{h,p}n^h < (1-\alpha)(1+\tau^h)n^h$. Using Lemma A8, we get $b - \underline{b} - w^b n - \overline{n} < (1-\alpha)(1+\tau^h)n^h + w^p n$. Furthermore, if $r^{pt} < r^d$, then $w^p \leq 0$ by Equations 19, 21, and 22. Thus, $b - \underline{b} - w^b n < \overline{n} + (1-\alpha)(1+\tau^h)n^h$.

Assume that $b - \underline{b} - w^b n < \overline{n} + (1-\alpha)(1+\tau^h)n^h$. Using Lemma A8, we get $w^{h,p}n^h + w^p n < (1-\alpha)(1+\tau^h)n^h$. We have two cases.

- Case $w^p \geq 0$. Then, $w^{h,p}n^h < (1-\alpha)(1+\tau^h)n^h$. Thus, by Equation 23, $r^{pt} < r^d$.
- Case $w^p < 0$. Then, $\vartheta^m = 0$ and $r^k = r^m > r^p$ and $r^{pt} < r^d$.

□

B.2 Proof of Lemma 3

Proof. If $\vartheta^m > 0$, then $w^p = \kappa w^m > 0$. Thus, by Equation 20 and 21, $r^p > r^m$. If $r^p > r^m$, by Equation 20 and 21, $\vartheta^m > 0$.

Thus, given Lemma A 9, $r^p > r^m$ if and only if $b - \underline{b} - w^b n > \overline{n} + (1-\alpha)(1+\tau^h)n^h + \kappa m$.

Lemma A10 provides the proof of the second inequality. □

B.3 Proof of Lemma 4

Assume by way of contradiction that $r^{rrp} > r^{pt}$. Then households have a better investment opportunity than the market rate for triparty repo.

Assume by way of contradiction that $r^{rp} < r^{pt}$ with a repo facility open only to traditional banks. Then traditional bank dealers have a cheaper funding option available than the triparty repo rate.

Assume by way of contradiction that $r^{rp} < r^p$ with a broad access repo facility. Then shadow banks have a better funding rate available than the market rate of bilateral repo.

Assume $rp = rrp = 0$. Then,

$$w^h n^h = b - \underline{b} - w^b n - \overline{n} - w^p n + rrp - rp \tag{175}$$
$$= b - \underline{b} - \kappa(\underline{b} - a) - \overline{n} - w^p n + rrp - rp \tag{176}$$
$$= b - (1+\kappa)\underline{b} + \kappa a - \overline{n} - w^p n \tag{177}$$
$$\tag{178}$$

B.4 Proof of Proposition 3

From the traditional bank first-order conditions, we have that

$$r^p = \begin{cases} r^m + \kappa\vartheta^m - \chi\ell & \text{if } w_t^p < 0, \\ \in [r^m + \kappa\vartheta^m - \chi\ell, r^m + (1+\kappa)\vartheta^m] & \text{if } w_t^p = 0, \\ r^m + (1+\kappa)\vartheta^m & \text{if } w_t^p > 0. \end{cases} \tag{179}$$

It is direct to see that if $\vartheta^m = 0$, then $r^p \le r^m$. Thus, we only consider cases with $\vartheta^m > 0$. Then $w^p n = \kappa m > 0$ and $r^p = \min\{r^{rp}, r^m + (1+\kappa)\vartheta^m\}$.

Assume $\chi = 0$ and $\vartheta^m > 0$. From the traditional bank first-order conditions, we have $r^p = r^{pt} = r^d + \vartheta^m > r^d = r^k = r^m + \kappa\vartheta^m$. Thus, $r^{pt} > r^m > r^{rrp}$ and $rrp = 0$. For markets to clear, we need $\mathcal{W}(r^{pt} - r^d, \alpha) = b - \underline{b} - \kappa(\underline{b} - a) - \overline{n} - rp$. Since $rp \ge 0$, $\mathcal{W}(r^{pt} - r^d, \alpha) \le b - \underline{b} - \kappa(\underline{b} - a) - \overline{n}$. Given that $r^{pt} > r^d$, $\mathcal{W}(r^{pt} - r^d, \alpha) > \mathcal{W}(0, \alpha) = (1-\alpha)(1+\tau^h)n^h$, and we have a contradiction.

Table 2: Repo Dynamics Summary Regression The table presents results from daily regressions of the first-difference repo to interest on reserves spreads ($\Delta TGCF - IOR$), the repo intermediation spreads between inter-dealer repo and dealer-to-money-fund repo ($\Delta TGCF - TGCR$) as well as Quantities at the Reverse Repo Facility (ΔRRP), and balance on the Treasury General Account (ΔTGA) on dummy variables indicating if the day is the last day of a quarter ($Quarter\ End$), the first day of a quarter ($Quarter\ End + 1$), a tax deadline day ($Tax\ Deadline$), and a day following a tax deadline day ($Tax\ Deadline + 1$) as well as a continuous variable indicating the daily change in Treasury outstanding ($\Delta Treasury\ Issuance$).

	(1) Δ TGCF-IOR	(2) Δ TGCF-TGCR	(3) Δ RRP	(4) Δ TGA
Quarter End	9.625***	0.0693***	102.4***	29.53***
	(1.967)	(0.017)	(12.117)	(4.732)
Quarter End +1	-5.940	-0.0464	-118.7***	-40.93***
	(4.111)	(0.028)	(18.006)	(5.245)
Tax Deadline	2.739***	0.00929	-0.304	47.10***
	(0.446)	(0.006)	(3.002)	(6.614)
Tax Deadline +1	4.547	-0.00979	11.02***	14.57***
	(6.230)	(0.012)	(2.544)	(3.066)
Δ Treasury Issuance	0.0165***	0.0000882***	0.00256	0.0417***
	(0.003)	(0.000)	(0.008)	(0.007)
Constant	-0.275	0.000300	-0.511	-4.056***
	(0.210)	(0.001)	(0.499)	(0.363)
Observations	2,010	1,971	1,277	2,010

*p<0.05, ** p<0.01, *** p<0.001

C Relegated Table